1.

次の積分を計算せよ.

(1) $\displaystyle\int x^4 dx$　　(2) $\displaystyle\int \frac{1}{x^2}dx$

(3) $\displaystyle\int_1^e \frac{dx}{x}$　　(4) $\displaystyle\int_1^2 \sqrt{x}dx$

(5) $\displaystyle\int_1^8 \sqrt[3]{x}dx$　　(6) $\displaystyle\int_1^4 \frac{dt}{\sqrt{t}}$

[解]

(1) $\displaystyle\int x^4 dx = \frac{x^5}{5} + C$ // 以下Cは積分定数とする

(2) $\displaystyle\int \frac{1}{x^2}dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$ //

(3) $\displaystyle\int_1^e \frac{dx}{x} = \Big[\log x\Big]_1^e$ ← $1 \leqq x \leqq e$ より $|x| = x$

$= \log e - \log 1 = 1$ //

(4) $\displaystyle\int_1^2 \sqrt{x}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2$

$= \frac{2}{3}(2\sqrt{2} - 1)$ //

(5) $\displaystyle\int_1^8 \sqrt[3]{x}\,dx = \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^8$

$= \frac{3}{4}(16 - 1) = \frac{45}{4}$ //

(6) $\displaystyle\int_1^4 \frac{dt}{\sqrt{t}} = \left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$

$= 2(2 - 1) = 2$ //

〈point〉

① 不定積分

関数 $f(x)$ に対して微分すると $f(x)$ になる関数を $f(x)$ の不定積分または原始関数といい, $\int f(x)dx$ と表す.

$f(x)$ の不定積分の1つを $F(x)$ とすると, $f(x)$ の不定積分は

$$\int f(x)dx = F(x) + C \quad (C: 積分定数)$$

と表される.

($f(x)$ の不定積分は1つに定まらない(不定)が *ちがいは定数の部分だけなので $F(x)$ を1つ見つけ C をつけるということ)

*について

$F(x)$, $G(x)$ を $f(x)$ の不定積分とすると

$$(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$$

$\therefore F(x) - G(x) = C$　　$\therefore F(x) = G(x) + C$ 〼

② x^α の積分

① $\displaystyle\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C \quad (\alpha \in \mathbb{R}, \alpha \neq -1)$

② $\displaystyle\int \frac{1}{x}dx = \log|x| + C$

[証明]

① $\displaystyle\left(\frac{1}{\alpha+1}x^{\alpha+1}\right)' = x^\alpha$

② $\displaystyle(\log|x|)' = \frac{1}{x}$

2.

次の不定積分を求めよ．

(1) $\displaystyle\int \frac{2x+1}{x^2}dx$　　(2) $\displaystyle\int \frac{(\sqrt{x}-1)^2}{x}dx$

[解]

(1) $\displaystyle\int \frac{2x+1}{x^2}dx = \int \left(\frac{2}{x}+\frac{1}{x^2}\right)dx$

$\displaystyle = 2\log|x| - \frac{1}{x} + C$

(2) $\displaystyle\int \frac{(\sqrt{x}-1)^2}{x}dx = \int \frac{x-2\sqrt{x}+1}{x}dx$

$\displaystyle = \int \left(1-\frac{2}{\sqrt{x}}+\frac{1}{x}\right)dx$

$= x - 4\sqrt{x} + \log|x| + C$

<point>

① 線形性

(1) $\int k f(x)\,dx = k\int f(x)\,dx$ $(k\in\mathbb{R})$ (定数倍)

(2) $\int \{f(x)\pm g(x)\}dx = \int f(x)\,dx \pm \int g(x)\,dx$ (和)

[証明]

(1) $(k\int f(x)\,dx)' = k f(x)$

(2) $(\int f(x)\,dx \pm \int g(x)\,dx)' = f(x)\pm g(x)$

注．

積分には積や商の積分法はない

(近いものはある)．

3.

次の積分を計算せよ.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x dx$　　(2) $\displaystyle\int (\sin x + 2\cos x)dx$

(3) $\displaystyle\int \frac{\sin^2 x}{1+\cos x}dx$　　(4) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^2\theta}$

(5) $\displaystyle\int \tan^2 x dx$　　(6) $\displaystyle\int \frac{\sin^3 x - 1}{\sin^2 x}dx$

[解]

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}}$
$= -(0-1)$
$= 1$ //

(2) $\displaystyle\int (\sin x + 2\cos x)dx = -\cos x + 2\sin x + C$ //

(3) $\displaystyle\int \frac{\sin^2 x}{1+\cos x}dx = \int \frac{1-\cos^2 x}{1+\cos x}dx$
$\displaystyle= \int (1-\cos x)dx$
$= x - \sin x + C$ //

(4) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^2\theta} = \left[\tan\theta\right]_0^{\frac{\pi}{4}}$
$= \tan\frac{\pi}{4} - \tan 0$
$= 1 - 0 = 1$ //

(5) $\displaystyle\int \tan^2 x\, dx = \int \left(\frac{1}{\cos^2 x} - 1\right)dx \leftarrow \frac{1}{\cos^2\theta} = 1 + \tan^2\theta$
$= \tan x - x + C$ //

(6) $\displaystyle\int \frac{\sin^3 x - 1}{\sin^2 x}dx = \int \left(\sin x - \frac{1}{\sin^2 x}\right)dx$
$\displaystyle= -\cos x + \frac{1}{\tan x} + C$ //

<point>

① 三角関数の積分

③ $\displaystyle\int \sin x\, dx = -\cos x + C$

④ $\displaystyle\int \cos x\, dx = \sin x + C$

⑤ $\displaystyle\int \frac{1}{\cos^2 x}dx = \tan x + C$

⑥ $\displaystyle\int \frac{1}{\sin^2 x}dx = -\frac{1}{\tan x} + C$

[証明]

③ $(-\cos x)' = \sin x$

④ $(\sin x)' = \cos x$

⑤ $\displaystyle(\tan x)' = \frac{1}{\cos^2 x}$

⑥ $\displaystyle\left(-\frac{1}{\tan x}\right)' = \frac{1}{\tan^2 x}\cdot\frac{1}{\cos^2 x} = \frac{1}{\sin^2 x}$ □

4.

次の不定積分を求めよ．

(1) $\displaystyle\int (3e^x - x^3)dx$ (2) $\displaystyle\int (3^x + 5^x)dx$

[解]

(1) $\displaystyle\int (3e^x - x^3)\,dx = 3e^x - \frac{x^4}{4} + C$ //

(2) $\displaystyle\int (3^x + 5^x)\,dx = \frac{3^x}{\log 3} + \frac{5^x}{\log 5} + C$ //

〈point〉

① 指数関数の積分

㋐ $\displaystyle\int e^x\,dx = e^x + C$

㋑ $\displaystyle\int a^x\,dx = \frac{a^x}{\log a} + C$

[証明]

㋐ $(e^x)' = e^x$

㋑ $\displaystyle\left(\frac{a^x}{\log a}\right)' = \frac{1}{\log a}\cdot a^x\log a = a^x$ 口

5.

次の積分を計算せよ．

(1) $\int (2x-5)^3 dx$　　(2) $\int \frac{dx}{(1-x)^2}$

(3) $\int \frac{2}{2x+1} dx$　　(4) $\int_0^{\frac{1}{2}} \sqrt{1-2x}\,dx$

(5) $\int_0^{\pi} \sin 3x\,dx$　　(6) $\int_0^1 e^{2x} dx$

[解]

(1) $\int (2x-5)^3 dx = \frac{(2x-5)^4}{4\cdot 2} + C$

$= \frac{(2x-5)^4}{8} + C$ //

(2) $\int \frac{dx}{(1-x)^2} = \frac{(1-x)^{-1}}{-1\cdot(-1)} + C$

$= \frac{1}{1-x} + C$ //

(3) $\int \frac{2}{2x+1} dx = 2\cdot\frac{\log|2x+1|}{2} + C$

$= \log|2x+1| + C$ //

(4) $\int_0^{\frac{1}{2}} \sqrt{1-2x}\,dx = \left[\frac{(1-2x)^{\frac{3}{2}}}{\frac{3}{2}\cdot(-2)}\right]_0^{\frac{1}{2}}$

$= -\frac{1}{3}(0-1) = \frac{1}{3}$ //

(5) $\int_0^{\pi} \sin 3x\,dx = \left[\frac{-\cos 3x}{3}\right]_0^{\pi}$

$= -\frac{1}{3}(-1-1)$

$= \frac{2}{3}$ //

(6) $\int_0^1 e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^1$

$= \frac{1}{2}(e^2-1)$ //

<point>

① $F(x) = \int f(x)\,dx$ とするとき，($F'(x) = f(x)$)

$\int f(\underbrace{ax+b}_{1次式})\,dx = \frac{1}{a}F(ax+b) + C$

[証明]

$\left(\frac{1}{a}F(ax+b)\right)' = \frac{1}{a}F'(ax+b)\cdot a$

$= f(ax+b)$ ■

6.

次の不定積分を求めよ．

(1) $\displaystyle\int 2x\sqrt{x^2+1}dx$　(2) $\displaystyle\int \cos^2 x \sin x dx$　(3) $\displaystyle\int \frac{\cos x}{\sin^2 x}dx$

(4) $\displaystyle\int \frac{dx}{1-\sin x}$　(5) $\displaystyle\int xe^{x^2}dx$　(6) $\displaystyle\int \frac{\log x}{x}dx$

[解]

(1) $\displaystyle\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\,(x^2+1)'dx$

$\displaystyle= \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$

$\displaystyle= \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$ 〃

(2) $\displaystyle\int \cos^2 x \sin x\, dx = -\int \cos^2 x\,(\cos x)'dx$

$\displaystyle= -\frac{\cos^3 x}{3} + C$ 〃

(3) $\displaystyle\int \frac{\cos x}{\sin^2 x}dx = \int (\sin x)^{-2}\cdot(\sin x)'dx$

$\displaystyle= \frac{(\sin x)^{-1}}{-1} + C$

$\displaystyle= -\frac{1}{\sin x} + C$ 〃

(4) $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{1+\sin x}{1-\sin^2 x}dx$

$\displaystyle= \int \frac{1+\sin x}{\cos^2 x}dx$

$\displaystyle= \int \left(\frac{1}{\cos^2 x} - (\cos x)^{-2}\cdot(\cos x)'\right)dx$

$\displaystyle= \tan x + \frac{1}{\cos x} + C$ 〃

(5) $\displaystyle\int xe^{x^2}dx = \frac{1}{2}\int e^{x^2}\cdot(x^2)'dx$

$\displaystyle= \frac{1}{2}e^{x^2} + C$ 〃

(6) $\displaystyle\int \frac{\log x}{x}dx = \int \log x\cdot(\log x)'dx$

$\displaystyle= \frac{(\log x)^2}{2} + C$ 〃

〈point〉

① $F(x) = \int f(x)\,dx$ とするとき　$(F'(x) = f(x))$

$\displaystyle\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$

[証明]

$(F(g(x)))' = F'(g(x))\cdot g'(x)$

$= f(g(x))\,g'(x)$

7.

次の不定積分を求めよ．

(1) $\displaystyle\int \sin^3 x\,dx$　　　(2) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

[解]

(1) $\displaystyle\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx$

$\displaystyle= \int (1-\cos^2 x)\sin x\,dx$

$\displaystyle= \int (\sin x + \cos^2 x\,(\cos x)')\,dx$

$\displaystyle= -\cos x + \frac{\cos^3 x}{3} + C$ 〃

(2) $\displaystyle\int \sin^3 x \cos^2 x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx$

$\displaystyle= \int (-\cos^2 x \cdot (\cos x)' + \cos^4 x\,(\cos x)')\,dx$

$\displaystyle= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$ 〃

〈point〉

① sin, cos の奇数乗の積分

8.

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{2x}{x^2+4}dx$　　(2) $\displaystyle\int \frac{e^x}{e^x+1}dx$　　(3) $\displaystyle\int \tan x dx$

[解]

(1) $\displaystyle\int \frac{2x}{x^2+4}dx = \int \frac{(x^2+4)'}{x^2+4}dx$

$= \log(x^2+4) + C$ // ← $x^2+4>0$

(2) $\displaystyle\int \frac{e^x}{e^x+1}dx = \int \frac{(e^x+1)'}{e^x+1}dx$

$= \log(e^x+1) + C$ // ← $e^x+1>0$

(3) $\displaystyle\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$

$\displaystyle = -\int \frac{(\cos x)'}{\cos x}dx$

$= -\log|\cos x| + C$ //

<point>

① $\displaystyle\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C.$

[証明]

$\displaystyle(\log|f(x)|)' = \frac{1}{f(x)}\cdot f'(x)$ □.

② ① $\displaystyle\int \tan x\, dx = -\log|\cos x| + C$

(覚える必要はない)

9.

自然数 n について数列 $\{a_n\}$ を

$$a_n = \int_0^{\frac{\pi}{2}} (\sin 2x - 2\sin x)\left(\sum_{k=1}^{n} \cos^{k-1} x\right) dx$$

と定めるとき，$\lim_{n\to\infty} a_n$ を求めよ．

（関西大）

[解]

$$a_n = \int_0^{\frac{\pi}{2}} (\sin 2x - 2\sin x)\left(\sum_{k=1}^{n} \cos^{k-1} x\right) dx$$

被積分関数は $x=0$ のときは

$0 \cdot n = 0$

$0 < x \leqq \frac{\pi}{2}$ のときは

$$-2\sin x(\cancel{1-\cos x}) \cdot \frac{1-\cos^n x}{\cancel{1-\cos x}}$$

$$= -2\sin x(1-\cos^n x)$$

$$= 2(\cos^n x - 1)\sin x$$

これは $x=0$ のときも満たす．

$$a_n = \int_0^{\frac{\pi}{2}} 2(\cos^n x - 1)\sin x\, dx$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos^n x \sin x - \sin x)\, dx$$

$$= 2\left[-\frac{1}{n+1}\cos^{n+1} x + \cos x\right]_0^{\frac{\pi}{2}}$$

$$= 2\left(\frac{1}{n+1} - 1\right)$$

$$\lim_{n\to\infty} a_n = -2 \,/\!/$$

[別解]

$$a_n = \int_0^{\frac{\pi}{2}} 2\sin x(\cos x - 1)\sum_{k=1}^{n}\cos^{k-1} x\, dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin x \sum_{k=1}^{n}(\cos^k x - \cos^{k-1} x)\, dx$$

ここで

$$\sum_{k=1}^{n}(\cos^k x - \cos^{k-1} x)$$

$$= (\cancel{\cos^2 x} - \cos x) + (\cancel{\cos^3 x} - \cancel{\cos^2 x})$$
$$+ \cdots + (\cos^n x - \cancel{\cos^{n-1} x})$$

$$= \cos^n x - \cos x$$

あとは同様