1.

次の不定積分を求めよ．

(1) $\displaystyle\int \frac{2x^2-1}{x+1}dx$　　　(2) $\displaystyle\int \frac{x^2+x+1}{x^2+1}dx$

[解]

(1) $\displaystyle\int \frac{2x^2-1}{x+1}dx$

$$\begin{array}{r} 2x-2 \\ x+1 \,\big)\, \overline{2x^2 \qquad -1} \\ \underline{2x^2+2x} \\ -2x-1 \\ \underline{-2x-2} \\ 1 \end{array}$$

$\displaystyle\frac{2x^2-1}{x+1} = \frac{(x+1)(2x-2)+1}{x+1}$

$\displaystyle = 2x-2+\frac{1}{x+1}$ より

$\displaystyle = \int \left(2x-2+\frac{1}{x+1}\right)dx$

$= x^2-2x+\log|x+1|+C$ //

(2) $\displaystyle\int \frac{x^2+x+1}{x^2+1}dx$

$\displaystyle = \int \left(1+\frac{x}{x^2+1}\right)dx$

$\displaystyle = \int \left(1+\frac{1}{2}\cdot\frac{(x^2+1)'}{x^2+1}\right)dx$

$\displaystyle = x+\frac{1}{2}\log(x^2+1)+C$ //

〈point〉

① 分数関数の積分 1.

分数関数は 分子の次数 > 分母の次数のときは必ず分子の次数 < 分母の次数となるように変形する.

2.

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{x-3}{(x-1)(x-2)}dx$ (2) $\displaystyle\int_2^3 \frac{dx}{x^2-1}$

(3) $\displaystyle\int \frac{1}{x(x+1)(x+2)}dx$ (4) $\displaystyle\int \frac{x^3}{x^2+x-2}dx$

[解].

(1) $I=\displaystyle\int \frac{x-3}{(x-1)(x-2)}dx$ とおく

$\dfrac{x-3}{(x-1)(x-2)}=\dfrac{a}{x-1}+\dfrac{b}{x-2}$ とおく

$=\dfrac{a(x-2)+b(x-1)}{(x-1)(x-2)}$

$=\dfrac{(a+b)x-2a-b}{(x-1)(x-2)}$

$\begin{cases} a+b=1 \\ -2a-b=-3 \end{cases}$ $\therefore a=2,\ b=-1$

$\therefore \dfrac{x-3}{(x-1)(x-2)}=\dfrac{2}{x-1}-\dfrac{1}{x-2}$

$I=\displaystyle\int\left(\frac{2}{x-1}-\frac{1}{x-2}\right)dx$

$=2\log|x-1|-\log|x-2|+C$

$=\log\dfrac{(x-1)^2}{|x-2|}+C$ //

(2) $I=\displaystyle\int_2^3\frac{dx}{x^2-1}$ とおく

$\dfrac{1}{x^2-1}=\dfrac{a}{x+1}+\dfrac{b}{x-1}$ とおく.

$=\dfrac{a(x-1)+b(x+1)}{x^2-1}$

$=\dfrac{(a+b)x-a+b}{x^2-1}$

$\begin{cases} a+b=0 \\ -a+b=1 \end{cases}$ $\therefore a=-\dfrac{1}{2},\ b=\dfrac{1}{2}$

$\therefore \dfrac{1}{x^2-1}=\dfrac{1}{2}\left(\dfrac{1}{x-1}-\dfrac{1}{x+1}\right)$

$I=\dfrac{1}{2}\displaystyle\int_2^3\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

$=\dfrac{1}{2}\Big[\log|x-1|-\log|x+1|\Big]_2^3$

$=\dfrac{1}{2}\Big[\log\left|\dfrac{x-1}{x+1}\right|\Big]_2^3$

$=\dfrac{1}{2}\left(\log\dfrac{1}{2}-\log\dfrac{1}{3}\right)=\dfrac{1}{2}\log\dfrac{3}{2}$ //

(3) $I=\displaystyle\int\frac{1}{x(x+1)(x+2)}dx$ とおく

$\dfrac{1}{x(x+1)(x+2)}=\dfrac{a}{x}+\dfrac{b}{x+1}+\dfrac{c}{x+2}$ とおく

$=\dfrac{a(x+1)(x+2)+bx(x+2)+cx(x+1)}{x(x+1)(x+2)}$

$=\dfrac{(a+b+c)x^2+(3a+2b+c)x+2a}{x(x+1)(x+2)}$

$\begin{cases} a+b+c=0 \\ 3a+2b+c=0 \\ 2a=1 \end{cases}$ $\therefore a=\dfrac{1}{2},\ b=-1,\ c=\dfrac{1}{2}$

$\therefore \dfrac{1}{x(x+1)(x+2)}=\dfrac{1}{2}\left(\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}\right)$

$I=\dfrac{1}{2}\displaystyle\int\left(\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}\right)dx$

$=\dfrac{1}{2}\left(\log|x|-2\log|x+1|+\log|x+2|\right)+C$

$=\dfrac{1}{2}\log\dfrac{|x(x+2)|}{(x+1)^2}+C$ //

(4) $I=\displaystyle\int\frac{x^3}{x^2+x-2}dx$

$\dfrac{x^3}{x^2+x-2}=\dfrac{(x^2+x-2)(x-1)+3x-2}{x^2+x-2}$

$=x-1+\dfrac{3x-2}{x^2+x-2}$

$$\begin{array}{r} x-1 \\ x^2+x-2\,\overline{)\,x^3} \\ \underline{x^3+x^2-2x} \\ -x^2+2x \\ \underline{-x^2-x+2} \\ 3x-2 \end{array}$$

$$\frac{3x-2}{x^2+x-2} = \frac{a}{x+2} + \frac{b}{x-1} \text{ とおく}$$

$$= \frac{a(x-1)+b(x+2)}{x^2+x-2}$$

$$= \frac{(a+b)x-a+2b}{x^2+x-2}$$

$$\begin{cases} a+b=3 \\ -a+2b=-2 \end{cases} \qquad \therefore\ a=\frac{8}{3},\ b=\frac{1}{3}$$

$$\therefore\ \frac{x^3}{x^2+x-2} = \frac{1}{3}\left(3x-3+\frac{8}{x+2}+\frac{1}{x-1}\right)$$

$$I = \frac{1}{3}\int\left(3x-3+\frac{8}{x+2}+\frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}\left(\frac{3}{2}x^2-3x+8\log|x+2|+\log|x-1|\right)+C$$

$$= \frac{1}{2}x^2-x+\frac{1}{3}\log(x+2)^8|x-1|+C$$

〈point〉

① 分数関数の積分2.
部分分数分解.

注.
積分には(一般的な)積の積分法はないので
積分は積に弱い
→ 和に直す

3.

次の積分を計算せよ.

(1) $\displaystyle\int \frac{dx}{x(x^2+1)}$　　(2) $\displaystyle\int \frac{4x^2+x+1}{x^3-1}dx$

(3) $\displaystyle\int_0^1 \frac{2x+1}{(x+1)^2(x-2)}dx$

[解].

(1) $I=\displaystyle\int \frac{dx}{x(x^2+1)}$ とおく.

$\dfrac{1}{x(x^2+1)}=\dfrac{a}{x}+\dfrac{bx+c}{x^2+1}$ とおく

$=\dfrac{a(x^2+1)+(bx+c)x}{x(x^2+1)}$

$=\dfrac{(a+b)x^2+cx+a}{x(x^2+1)}$

$\begin{cases} a+b=0 \\ c=0 \\ a=1 \end{cases}$　$\therefore a=1,\ b=-1,\ c=0$

$\therefore \dfrac{1}{x(x^2+1)}=\dfrac{1}{x}-\dfrac{x}{x^2+1}$

$I=\displaystyle\int\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$

$=\displaystyle\int\left(\frac{1}{x}-\frac{1}{2}\cdot\frac{(x^2+1)'}{x^2+1}\right)dx$

$=\log|x|-\dfrac{1}{2}\log(x^2+1)+C$

$=\log\dfrac{|x|}{\sqrt{x^2+1}}+C$ //

(2) $I=\displaystyle\int \frac{4x^2+x+1}{x^3-1}dx$ とおく

$\dfrac{4x^2+x+1}{x^3-1}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+x+1}$ とおく

$=\dfrac{a(x^2+x+1)+(bx+c)(x-1)}{x^3-1}$

$=\dfrac{(a+b)x^2+(a-b+c)x+a-c}{x^3-1}$

$\begin{cases} a+b=4 \\ a-b+c=1 \\ a-c=1 \end{cases}$　$\therefore a=2,\ b=2,\ c=1$

$\therefore \dfrac{4x^2+x+1}{x^3-1}=\dfrac{2}{x-1}+\dfrac{2x+1}{x^2+x+1}$

$I=\displaystyle\int\left(\frac{2}{x-2}+\frac{2x+1}{x^2+x+1}\right)dx$

$=\displaystyle\int\left(\frac{2}{x-2}+\frac{(x^2+x+1)'}{x^2+x+1}\right)dx$

$=2\log|x-2|+\log(x^2+x+1)+C$

$=\log(x-2)^2(x^2+x+1)+C$ //

(3) $I=\displaystyle\int_0^1 \frac{2x+1}{(x+1)^2(x-2)}dx$

$\dfrac{2x+1}{(x+1)^2(x-2)}=\dfrac{a}{x+1}+\dfrac{b}{(x+1)^2}+\dfrac{c}{x-2}$ とおく

$=\dfrac{a(x+1)(x-2)+b(x-2)+c(x+1)^2}{(x+1)^2(x-2)}$

$=\dfrac{(a+c)x^2+(-a+b+2c)x-2a-2b+c}{(x+1)^2(x-2)}$

$\begin{cases} a+c=0 \\ -a+b+2c=2 \\ -2a-2b+c=1 \end{cases}$　$\therefore a=-\dfrac{5}{9},\ b=\dfrac{1}{3},\ c=\dfrac{5}{9}$

$\therefore \dfrac{2x+1}{(x+1)^2(x-2)}=-\dfrac{1}{9}\left(\dfrac{5}{x+1}-\dfrac{3}{(x+1)^2}-\dfrac{5}{x-2}\right)$

$I=-\dfrac{1}{9}\displaystyle\int_0^1\left(\frac{5}{x+1}-\frac{3}{(x+1)^2}-\frac{5}{x-2}\right)dx$

$=-\dfrac{1}{9}\left[5\log|x+1|+\dfrac{3}{x+1}-5\log|x-2|\right]_0^1$

$=-\dfrac{1}{9}\left[5\log\left|\dfrac{x+1}{x-2}\right|+\dfrac{3}{x+1}\right]_0^1$

$=-\dfrac{1}{9}\left\{5\log 2+\dfrac{3}{2}-\left(5\log\dfrac{1}{2}+3\right)\right\}$

$=-\dfrac{10}{9}\log 2+\dfrac{1}{6}$ //

<point>

① (2) で

$$\frac{2x+1}{(x+1)^2(x-2)}=\frac{a}{x+1}+\frac{b}{(x+1)^2}+\frac{c}{x-2}$$

とおける理由.

原則通りやれば

$$\frac{2x+1}{(x+1)^2(x-2)}=\frac{mx+n}{(x+1)^2}+\frac{c}{x-2}$$

ここで

$$\frac{m(x+1)-m+n}{(x+1)^2}=\frac{m}{x+1}+\frac{-m+n}{(x+1)^2}$$

と変形できる.

あとは $m=a,\ -m+n=b$ とすればよい.

4.

関数 $f(x)=\dfrac{1}{x^3(1-x)}$ について，

(1) $f(x)=\dfrac{a_1}{x}+\dfrac{a_2}{x^2}+\dfrac{a_3}{x^3}+\dfrac{b}{1-x}$ とおいて，定数 a_1，a_2，a_3，b を求めよ．

(2) 不定積分 $\displaystyle\int f(x)dx$ を求めよ．

(3) 同様にして，不定積分 $\displaystyle\int \frac{dx}{x^p(1-x)}$ $(p=1,\ 2,\ 3,\ \cdots)$ を求めよ．

（神戸大）

［解］

(1) $$\frac{1}{x^3(1-x)}=\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\frac{b}{1-x}$$
$$=\frac{a_1x^2+a_2x+a_3}{x^3}+\frac{b}{1-x}$$
$$=\frac{(a_1x^2+a_2x+a_3)(1-x)+bx^3}{x^3(1-x)}$$
$$=\frac{(b-a_1)x^3+(a_1-a_2)x^2+(a_2-a_3)x+a_3}{x^3(1-x)}$$

$$\therefore \begin{cases} b-a_1=0 \\ a_1-a_2=0 \\ a_2-a_3=0 \\ a_3=1 \end{cases} \qquad \therefore a_1=a_2=a_3=b=1$$

(2) $$\int f(x)\,dx=\int\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{1-x}\right)dx$$
$$=\log|x|-\frac{1}{x}-\frac{1}{2x^2}-\log|1-x|$$
$$=\log\left|\frac{x}{1-x}\right|-\frac{1}{x}-\frac{1}{2x^2}+C$$

(3) $$\frac{1}{x^p(1-x)}=\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots+\frac{a_p}{x^p}+\frac{b}{1-x}$$
$$=\frac{a_1x^{p-1}+a_2x^{p-2}+\cdots+a_p}{x^p}+\frac{b}{1-x}$$
$$=\frac{(a_1x^{p-1}+a_2x^{p-2}+\cdots+a_p)(1-x)+bx^p}{x^p(1-x)}$$
$$=\frac{(b-a_1)x^p+(a_1-a_2)x^{p-1}+\cdots+(a_{p-1}-a_p)x+a_p}{x^p(1-x)}$$

$$\therefore \begin{cases} b-a_1=0 \\ a_1-a_2=0 \\ \quad\vdots \\ a_{p-1}-a_p=0 \\ a_p=1 \end{cases} \qquad \therefore a_1=a_2=\cdots=a_p=b=1$$

$$\int\frac{dx}{x^p(1-x)}$$
$$=\int\left(\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{x^p}+\frac{1}{1-x}\right)dx$$
$$=\log|x|-\frac{1}{x}-\frac{1}{2x^2}-\cdots-\frac{1}{(p-1)x^{p-1}}-\log|1-x|$$
$$=\log\left|\frac{x}{1-x}\right|-\frac{1}{x}-\frac{1}{2x^2}-\cdots-\frac{1}{(p-1)x^{p-1}}+C$$

5.

次の積分を計算せよ.

(1) $\displaystyle\int \sin^2 x dx$　　(2) $\displaystyle\int_{\frac{\pi}{6}}^{\pi} \sin x \cos x dx$

(3) $\displaystyle\int (\sin x - \cos x)^2 dx$　　(4) $\displaystyle\int \cos^4 x dx$

[解]

(1) $\displaystyle\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2} dx$ ← 半角の公式

$\displaystyle = \frac{x}{2} - \frac{\sin 2x}{4} + C$

(2) $\displaystyle\int_{\frac{\pi}{6}}^{\pi} \sin x \cos x\, dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi} \sin 2x\, dx$

$\displaystyle = \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{6}}^{\pi}$

$\displaystyle = -\frac{1}{4}\left(1-\frac{1}{2}\right)$

$\displaystyle = -\frac{1}{8}$ //

[別解]

$\displaystyle\int_{\frac{\pi}{6}}^{\pi} \sin x \cos x\, dx = \int_{\frac{\pi}{6}}^{\pi} \sin x (\sin x)' dx$

$\displaystyle = \left[\frac{\sin^2 x}{2}\right]_{\frac{\pi}{6}}^{\pi}$

$\displaystyle = \frac{1}{2}\left(0-\frac{1}{4}\right) = -\frac{1}{8}$ //

(3) $\displaystyle\int (\sin x - \cos x)^2 dx = \int (1 - 2\sin x \cos x)\, dx$

$\displaystyle = \int (1-\sin 2x)\, dx$

$\displaystyle = x + \frac{\cos 2x}{2} + C$ //

(4) $\displaystyle\int \cos^4 x\, dx = \int (\cos^2 x)^2 dx$

$\displaystyle = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$\displaystyle = \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx$

$\displaystyle = \frac{1}{4}\int \left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right) dx$

$\displaystyle = \frac{1}{4}\left(\frac{3}{2}x + 2\cdot\frac{\sin 2x}{2} + \frac{\sin 4x}{8}\right) + C$

$\displaystyle = \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$ //

〈point〉

① sin, cos の偶数乗の積分:
半角の公式を利用

6.

次の不定積分を求めよ.

(1) $\displaystyle\int \sin 2x \cos 3x dx$　　　(2) $\displaystyle\int \cos x \cos 3x dx$

[解]

(1) $I = \int \sin 2x \cos 3x \, dx$ とおく

$$\begin{array}{rl} \sin(3x+2x) = & \sin 3x \cos 2x + \cos 3x \sin 2x \\ -) \ \sin(3x-2x) = & \sin 3x \cos 2x - \cos 3x \sin 2x \\ \hline \sin 5x - \sin x = & 2\cos 3x \sin 2x \end{array}$$

$\therefore \sin 2x \cos 3x = \dfrac{1}{2}(\sin 5x - \sin x)$

$I = \dfrac{1}{2}\int(\sin 5x - \sin x)\,dx$

$= \dfrac{1}{2}\left(-\dfrac{\cos 5x}{5} + \cos x\right) + C$ //

(2) $I = \int \cos x \cos 3x \, dx$ とおく.

$$\begin{array}{rl} \cos(3x+x) = & \cos 3x \cos x - \sin 3x \sin x \\ +) \ \cos(3x-x) = & \cos 3x \cos x + \sin 3x \sin x \\ \hline \cos 4x + \cos 2x = & 2\cos 3x \cos x \end{array}$$

$\therefore \cos x \cos 3x = \dfrac{1}{2}(\cos 4x + \cos 2x)$

$I = \dfrac{1}{2}\int(\cos 4x + \cos 2x)\,dx$

$= \dfrac{1}{2}\left(\dfrac{\sin 4x}{4} + \dfrac{\sin 2x}{2}\right) + C$ //

7.

(1) $\displaystyle\int_0^{\pi} \cos^2 x dx$ を求めよ．

(2) $\displaystyle\int_0^{\pi} \cos mx \cos nx dx$ を求めよ．ただし，m，n は自然数とする．

(3) $\displaystyle\int_0^{\pi} \left(\sum_{k=1}^{n} \cos kx\right)^2 dx$ を求めよ．

（東北学院大）

[解]

(1) $\displaystyle\int_0^{\pi} \cos^2 x\, dx = \int_0^{\pi} \frac{1+\cos 2x}{2} dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

$\displaystyle = \frac{\pi}{2}$ //

(2) $\displaystyle I = \int_0^{\pi} \cos mx \cos nx\, dx$

(i) $m=n$ のとき

$\displaystyle I = \int_0^{\pi} \cos^2 mx\, dx$

$\displaystyle = \int_0^{\pi} \frac{1+\cos 2mx}{2} dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_0^{\pi}$

$\displaystyle = \frac{\pi}{2}$ //

(ii) $m \neq n$ のとき

$$\begin{aligned} &\cos(m+n)x = \overset{\circ}{\cos mx \cos nx} - \overset{\times}{\sin mx \sin nx} \\ +)\ &\cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx \\ \hline &\cos(m+n)x + \cos(m-n)x = 2\cos mx \cos nx \end{aligned}$$

$\displaystyle \therefore \cos mx \cos nx = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}$

$\displaystyle I = \frac{1}{2}\int_0^{\pi} \{\cos(m+n)x + \cos(m-n)x\} dx$

$\displaystyle = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi}$

$= 0.$

(3) $\displaystyle\int_0^{\pi} \left(\sum_{k=1}^{n} \cos kx\right)^2 dx$

$\displaystyle = \int_0^{\pi} (\cos x + \cos 2x + \cdots + \cos nx)^2 dx$

$\displaystyle\int_0^{\pi} \cos mx \cos nx\, dx = 0$ $(m \neq n)$ より

$\displaystyle = \int_0^{\pi} (\cos^2 x + \cos^2 2x + \cdots + \cos^2 nx) dx$

$\displaystyle\int_0^{\pi} \cos^2 mx\, dx = \frac{\pi}{2}$ より $(m \in \mathbb{N})$

$\displaystyle = \frac{n\pi}{2}$ //

8.

定積分 $I = \displaystyle\int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} d\theta$ を次の順に求めよう.

(1) $2\left(\sin\dfrac{\theta}{2}\right)\left(\dfrac{1}{2}+\cos\theta+\cos 2\theta+\cdots+\cos n\theta\right)$

$= \sin\dfrac{\theta}{2} + \displaystyle\sum_{k=1}^{n}\{\boxed{} - \boxed{}\} = \boxed{}$

(2) $\displaystyle\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta\right)d\theta = \boxed{}$

よって, (1) から $I = \boxed{}$ である. (芝浦工業大)

[解]

(1) $2\sin\frac{\theta}{2}\cos k\theta$ $(k=1, 2, \cdots, n)$

$\sin\left(k+\frac{1}{2}\right)\theta = \sin k\theta\cos\frac{\theta}{2} + \cos k\theta\sin\frac{\theta}{2}$

$-)\ \sin\left(k-\frac{1}{2}\right)\theta = \sin k\theta\cos\frac{\theta}{2} - \cos k\theta\sin\frac{\theta}{2}$

$\sin\left(k+\frac{1}{2}\right)\theta - \sin\left(k-\frac{1}{2}\right)\theta = 2\sin\frac{\theta}{2}\cos k\theta$

より

$2\left(\sin\frac{\theta}{2}\right)\left(\frac{1}{2}+\cos\theta+\cdots\cos n\theta\right)$

$= \sin\frac{\theta}{2} + \sum_{k=1}^{n}\left\{\boxed{\sin\left(k+\frac{1}{2}\right)\theta} - \boxed{\sin\left(k-\frac{1}{2}\right)\theta}\right\}$

$= \cancel{\sin\frac{\theta}{2}} + \left(\cancel{\sin\frac{3}{2}\theta} - \cancel{\sin\frac{\theta}{2}}\right) + \left(\cancel{\sin\frac{5}{2}\theta} - \cancel{\sin\frac{3}{2}\theta}\right)$

$+ \cdots + \left(\sin\left(n+\frac{1}{2}\right)\theta - \cancel{\sin\left(n-\frac{1}{2}\right)\theta}\right)$

$= \boxed{\sin\left(n+\frac{1}{2}\right)\theta}$

(2) $\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta\right)d\theta$

$= \int_{-\pi}^{\pi}\left(\frac{1}{2}+\cos\theta+\cos 2\theta+\cdots+\cos n\theta\right)d\theta$

$= 2\int_{0}^{\pi}\left(\frac{1}{2}+\cos\theta+\cos 2\theta+\cdots+\cos n\theta\right)d\theta$

$= 2\left[\frac{\theta}{2}+\sin\theta+\frac{\sin 2\theta}{2}+\cdots+\frac{\sin n\theta}{n}\right]_{0}^{\pi}$

$= \boxed{\pi}$

(1) から

$2\sin\frac{\theta}{2}\left(\frac{1}{2}+\cos\theta+\cdots+\cos n\theta\right) = \sin\left(n+\frac{1}{2}\right)\theta$

$\therefore \dfrac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos k\theta$

よって

$I = \int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta\right)d\theta$

$= \boxed{\pi}$