1.

次の積分を計算せよ.

(1) $\displaystyle\int xe^{2x}\,dx$　　(2) $\displaystyle\int_0^1 x^2e^x\,dx$　　(3) $\displaystyle\int (x-1)e^x\,dx$

((2) 立教大)

[解]

(1) $\displaystyle\int xe^{2x}dx$　　$\begin{array}{ll} f & \frac{e^{2x}}{2} \\ f' & e^{2x} \end{array}$　$\begin{array}{ll} g & x \\ g' & 1 \end{array}$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \ //$$

(2) $\displaystyle\int_0^1 x^2e^xdx$　　$\begin{array}{ll} f & e^x \\ f' & e^x \end{array}$　$\begin{array}{ll} g & x^2 \\ g' & 2x \end{array}$

$$= \left[x^2e^x\right]_0^1 - 2\int_0^1 xe^xdx$$

$$= e - 2\int_0^1 xe^xdx$$

ここで

$\displaystyle\int_0^1 xe^xdx$　　$\begin{array}{ll} f & e^x \\ f' & e^x \end{array}$　$\begin{array}{ll} g & x \\ g' & 1 \end{array}$

$$= \left[xe^x\right]_0^1 - \int_0^1 e^xdx$$

$$= e - \left[e^x\right]_0^1$$

$$= 1$$

よって

$$\int_0^1 x^2e^xdx = e - 2 \ //$$

(3) $\displaystyle\int (x-1)e^xdx$　　$\begin{array}{ll} f & e^x \\ f' & e^x \end{array}$　$\begin{array}{ll} g & x-1 \\ g' & 1 \end{array}$

$$= (x-1)e^x - \int e^xdx$$

$$= (x-1)e^x - e^x$$

$$= (x-2)e^x + C \ //$$

〈point〉

① 部分積分法.

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

[証明].

積の微分法より

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$$

$$f'(x)g(x) = \{f(x)g(x)\}' - f(x)g'(x)$$

両辺 x で積分して

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

注.

部分積分法は積の積分法にあたるものだがすべての積に有効なわけではない.

注.

$\int f(x)g'(x)dx$ がもとの $\int f'(x)g(x)dx$ より簡単になるように $f'(x)$, $g(x)$ をおく.

$f(x)$ は積分してもあまり変わらないもの

$g(x)$ は微分したら簡単になるものにするとよい。

2.

次の積分を計算せよ．

(1) $\displaystyle\int x\sin x dx$　　(2) $\displaystyle\int \frac{x}{\cos^2 x}dx$

(3) $\displaystyle\int_0^{\pi} x\sin^2 x dx$　　(4) $\displaystyle\int x^2 \sin x dx$

[解]

(1) $\displaystyle\int x\sin x\, dx$　　$f\ -\cos x$　$g\ x$　$f'\ \sin x$　$g'\ 1$

$= -x\cos x + \displaystyle\int \cos x\, dx$

$= -x\cos x + \sin x + C$ //

(2) $\displaystyle\int \frac{x}{\cos^2 x}dx$　　$f\ \tan x$　$g\ x$　$f'\ \frac{1}{\cos^2 x}$　$g'\ 1$

$= x\tan x - \displaystyle\int \tan x\, dx$

ここで

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{(\cos x)'}{\cos x}dx = -\log|\cos x|$$

より

$= x\tan x + \log|\cos x| + C$ //

(3) $$\int_0^{\pi} x\sin^2 x\, dx = \int_0^{\pi} x\cdot\frac{1-\cos 2x}{2}dx = \frac{1}{2}\int_0^{\pi}(x - x\cos 2x)dx$$

ここで

$$\int_0^{\pi} x\, dx = \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$$

$\displaystyle\int_0^{\pi} x\cos 2x$　　$f\ \frac{\sin 2x}{2}$　$g\ x$　$f'\ \cos 2x$　$g'\ 1$

$$= \frac{1}{2}[x\sin 2x]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin 2x\, dx$$

$$= -\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}(1-1) = 0$$

より

$$= \frac{1}{2}\left(\frac{\pi^2}{2} - 0\right) = \frac{\pi^2}{4}$$ //

(4) $\displaystyle\int x^2\sin x\, dx$　　$f\ -\cos x$　$g\ x^2$　$f'\ \sin x$　$g'\ 2x$

$= -x^2\cos x + 2\displaystyle\int x\cos x\, dx$

ここで

$\displaystyle\int x\cos x\, dx$　　$f\ \sin x$　$g\ x$　$f'\ \cos x$　$g'\ 1$

$= x\sin x - \displaystyle\int \sin x\, dx$

$= x\sin x + \cos x$

より

$= -x^2\cos x + 2(x\sin x + \cos x) + C$

$= (2 - x^2)\cos x + 2x\sin x + C$ //

3.

次の積分を計算せよ.

(1) $\displaystyle\int \log x\,dx$　　(2) $\displaystyle\int_0^1 x\log(x^2+1)\,dx$

(3) $\displaystyle\int_1^e x\log x\,dx$　　(4) $\displaystyle\int_1^e (\log x)^2\,dx$

[解]

(1) $\displaystyle\int \log x\,dx.$　　$\begin{array}{ll} f & x \\ f' & 1 \end{array}$　$\begin{array}{ll} g & \log x \\ g' & \frac{1}{x} \end{array}$

$\displaystyle = x\log x - \int 1\,dx$

$= x\log x - x + C$ // ← 公式

(2) $\displaystyle\int_0^1 x\log(x^2+1)\,dx$

$\displaystyle = \frac{1}{2}\int_0^1 \log(x^2+1)\cdot(x^2+1)'\,dx$

$\displaystyle = \frac{1}{2}\Big[(x^2+1)\log(x^2+1) - (x^2+1)\Big]_0^1$

$\displaystyle = \frac{1}{2}\{2\log 2 - 2 - (-1)\}$

$\displaystyle = \log 2 - \frac{1}{2}$ //

(3) $\displaystyle\int_1^e x\log x\,dx$　　$\begin{array}{ll} f & \frac{x^2}{2} \\ f' & x \end{array}$　$\begin{array}{ll} g & \log x \\ g' & \frac{1}{x} \end{array}$

$\displaystyle = \frac{1}{2}\Big[x^2\log x\Big]_1^e - \frac{1}{2}\int_1^e x\,dx$

$\displaystyle = \frac{e^2}{2} - \frac{1}{2}\Big[\frac{x^2}{2}\Big]_1^e$

$\displaystyle = \frac{e^2}{4} + \frac{1}{4}$ //

(4) $\displaystyle\int_1^e (\log x)^2\,dx$　　$\begin{array}{ll} f & x \\ f' & 1 \end{array}$　$\begin{array}{ll} g & (\log x)^2 \\ g' & 2\log x\cdot\frac{1}{x} \end{array}$

$\displaystyle = \Big[x(\log x)^2\Big]_1^e - 2\int_1^e \log x\,dx$

$= e - 2\Big[x\log x - x\Big]_1^e$

$= e - 2\cdot 1$

$= e - 2$ //

<point>

① ⑩ $\displaystyle\int \log x\,dx = x\log x - x + C$

(覚えた方がよい)

4.

次の定積分を求めよ．

(1) $\displaystyle\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx$　　　(2) $\displaystyle\int_{1}^{2}(x-1)(x-2)^3dx$

［解］

(1) $\displaystyle\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)\,dx$　　$f\ \dfrac{(x-\alpha)^2}{2}$　$g\ x-\beta$　$f'\ x-\alpha$　$g'\ 1$

$$=\left[\frac{(x-\alpha)^2}{2}(x-\beta)\right]_{\alpha}^{\beta}-\frac{1}{2}\int_{\alpha}^{\beta}(x-\alpha)^2dx$$

$$=-\frac{1}{2}\left[\frac{(x-\alpha)^3}{3}\right]_{\alpha}^{\beta}$$

$$=-\frac{1}{6}(\beta-\alpha)^3 〃$$

(2) $\displaystyle\int_{1}^{2}(x-1)(x-2)^3\,dx$　　$f\ \dfrac{(x-2)^4}{4}$　$g\ x-1$　$f'\ (x-2)^3$　$g'\ 1$

$$=\left[\frac{(x-2)^4}{4}\cdot(x-1)\right]_{1}^{2}-\frac{1}{4}\int_{1}^{2}(x-2)^4dx$$

$$=-\frac{1}{4}\left[\frac{(x-2)^5}{5}\right]_{1}^{2}$$

$$=-\frac{1}{20}(0+1)=-\frac{1}{20} 〃$$

5.

$I = \int e^x \sin x dx$, $J = \int e^x \cos x dx$ とする．I, J を計算せよ．

[解].

$$I = \int e^x \sin x \, dx \qquad \begin{matrix} f & -\cos x & g & e^x \\ f' & \sin x & g' & e^x \end{matrix}$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + J$$

$$\therefore I - J = -e^x \cos x \cdots ①$$

$$J = \int e^x \cos x \, dx \qquad \begin{matrix} f & \sin x & g & e^x \\ f' & \cos x & g' & e^x \end{matrix}$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x - I$$

$$\therefore I + J = e^x \sin x \cdots ②$$

①+② より

$$2I = e^x(\sin x - \cos x)$$

$$\therefore I = \frac{e^x}{2}(\sin x - \cos x) + C \,_{//}$$

②−① より

$$2J = e^x(\sin x + \cos x)$$

$$\therefore J = \frac{e^x}{2}(\sin x + \cos x) + C \,_{//}$$

[別解] 微分する

$$(e^x \sin x)' = e^x \sin x + e^x \cos x \cdots ①$$

$$(e^x \cos x)' = -e^x \sin x + e^x \cos x \cdots ②$$

①−② より

$$2e^x \sin x = (e^x \sin x)' - (e^x \cos x)'$$

$$e^x \sin x = \frac{1}{2}\{(e^x \sin x)' - (e^x \cos x)'\}$$

両辺 x で積分して

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C \,_{//}$$

①+② より

$$2e^x \cos x = (e^x \sin x)' + (e^x \cos x)'$$

$$e^x \cos x = \frac{1}{2}\{(e^x \sin x)' + (e^x \cos x)'\}$$

両辺 x で積分して

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C \,_{//}$$