1.

次の定積分を求めよ．

(1) $\displaystyle\int_0^1 x(1-x)^3dx$　　　(2) $\displaystyle\int_0^1 \frac{x-1}{(2-x)^2}dx$

［解］

(1) $I=\int_0^1 x(1-x)^3dx$ とおく

$t=1-x$ とおく

$\begin{cases} x: 0\to 1 \\ t: 1\to 0 \end{cases}$

$x=1-t$

$\dfrac{dx}{dt}=-1$

$I=\int_1^0 (1-t)t^3\cdot(-1)dt$

$=\int_0^1 (t^3-t^4)dt$

$=\left[\dfrac{t^4}{4}-\dfrac{t^5}{5}\right]_0^1=\dfrac{1}{20}$ //

［別解］

① 展開して積分

② 部分積分

③ $(x-1)$ のかたまりを作る

$I=-\int_0^1 x(x-1)^3dx$

$=-\int_0^1\{(x-1)+1\}(x-1)^3dx$

$=-\int_0^1\{(x-1)^4+(x-1)^3\}dx$

$=-\left[\dfrac{(x-1)^5}{5}+\dfrac{(x-1)^4}{4}\right]_0^1$

$=-\left\{0-\left(-\dfrac{1}{5}+\dfrac{1}{4}\right)\right\}=\dfrac{1}{20}$ //

(2) $I=\int_0^1\dfrac{x-1}{(2-x)^2}dx$ とおく

$t=2-x$ とおく

$\begin{cases} x: 0\to 1 \\ t: 2\to 1 \end{cases}$

$x=2-t$

$\dfrac{dx}{dt}=-1$

$I=\int_2^1\dfrac{1-t}{t^2}\cdot(-1)dt$

$=\int_1^2\left(\dfrac{1}{t^2}-\dfrac{1}{t}\right)dt$

$=\left[-\dfrac{1}{t}-\log|t|\right]_1^2$

$=-\dfrac{1}{2}-\log 2-(-1)=\dfrac{1}{2}-\log 2$ //

［別解］

① 部分分数分解

$\dfrac{x-1}{(2-x)^2}=\dfrac{a}{2-x}+\dfrac{b}{(2-x)^2}$ と変形

〈point〉

① 置換積分法

x が微分可能な t の関数 $g(t)$ を用いて

$x=g(t)$ と表されるとき

$\int f(x)dx=\int f(g(t))g'(t)dt$

$=\int f(g(t))\dfrac{dx}{dt}dt$

［証明］

$F(x)=\int f(x)dx$ とする（$F'(x)=f(x)$）

$\dfrac{d}{dt}F(g(t))=F'(g(t))\cdot g'(t)$

$=f(g(t))\cdot g'(t)$ であるから

$\int f(g(t))g'(t)dt=F(g(t))$

$x=g(t)$ より

$=F(x)$

$=\int f(x)dx$ □

注

$x=g(t)$, $dx=\dfrac{dx}{dt}dt$ と考えるとよい

↑分数のように思う

2.

次の積分を計算せよ.

(1) $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$　　(2) $\displaystyle\int \frac{x}{\sqrt{3x-1}}dx$

(3) $\displaystyle\int \frac{dx}{x\sqrt{x+1}}$　　(4) $\displaystyle\int_0^4 \sqrt{2-\sqrt{x}}\,dx$　　((4) 小樽商科大)

[解]

(1) $I=\int_0^1 x\sqrt{1-x}\,dx$ とおく.

$t=1-x$ とおく

$\begin{cases} x: 0\to 1 \\ t: 1\to 0 \end{cases}$

$x=1-t$

$\frac{dx}{dt}=-1$

$I=\int_1^0 (1-t)\sqrt{t}\cdot(-1)\,dt$

$=\int_0^1 (t^{\frac{1}{2}}-t^{\frac{3}{2}})\,dt$

$=\left[\frac{2}{3}t^{\frac{3}{2}}-\frac{2}{5}t^{\frac{5}{2}}\right]_0^1$

$=\frac{2}{3}-\frac{2}{5}=\frac{4}{15}$ //

[別解]

① $t=\sqrt{1-x}$ とおく

$\begin{cases} x: 0\to 1 \\ t: 1\to 0 \end{cases}$

$t^2=1-x,\ t\geqq 0$

$\therefore x=1-t^2$

$\frac{dx}{dt}=-2t$

$I=\int_1^0 (1-t^2)t\cdot(-2t)\,dt$

$=2\int_0^1 (t^2-t^4)\,dt$

$=2\left[\frac{t^3}{3}-\frac{t^5}{5}\right]_0^1=\frac{4}{15}$ //

② 部分積分

③ $I=-\int -x\sqrt{1-x}\,dx$

$=-\int \{(1-x)-1\}\sqrt{1-x}\,dx$

$=-\int \{(1-x)^{\frac{3}{2}}-(1-x)^{\frac{1}{2}}\}\,dx$

$=\frac{4}{15}$ //

(2) $I=\int \frac{x}{\sqrt{3x-1}}dx$ とおく

$t=3x-1$ とおく　($t=\sqrt{3x-1}$ とおいてもよい)

$x=\frac{t}{3}+\frac{1}{3}$

$\frac{dx}{dt}=\frac{1}{3}$

$I=\int \frac{\frac{t}{3}+\frac{1}{3}}{\sqrt{t}}\cdot\frac{1}{3}\,dt$

$=\frac{1}{9}\int (t^{\frac{1}{2}}+t^{-\frac{1}{2}})\,dt$

$=\frac{1}{9}\left(\frac{2}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right)+C$

$=\frac{2}{27}\sqrt{3x-1}(3x+2)+C$ //

(3) $I=\int \frac{dx}{x\sqrt{x+1}}$ とおく

$t=\sqrt{x+1}$ とおく　($t=x+1$ ではうまくいかない)

$t^2=x+1,\ t\geqq 0$

$x=t^2-1$

$\frac{dx}{dt}=2t$

$I=\int \frac{1}{(t^2-1)\cancel{t}}\cdot 2\cancel{t}\,dt$

$=2\int \frac{1}{t^2-1}\,dt$

$\frac{1}{t^2-1}=\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$

$I=\int \left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$

$=\log|t-1|-\log|t+1|+C$

$=\log\left|\frac{t-1}{t+1}\right|+C$

$=\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$ //

(4) $I = \int_0^4 \sqrt{2-\sqrt{x}}\,dx$ とおく

$t = 2-\sqrt{x}$ とおく $\quad \begin{cases} x : 0 \to 4 \\ t : 2 \to 0 \end{cases}$

$\sqrt{x} = 2-t$

$x = (2-t)^2,\ t \leqq 2.$

$\quad = t^2-4t+4$

$\frac{dx}{dt} = 2t-4.$

$I = \int_2^0 \sqrt{t}\,(2t-4)\,dt$

$\quad = 2\int_0^2 (2t^{\frac{1}{2}} - t^{\frac{3}{2}})\,dt$

$\quad = 2\left[\frac{4}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}}\right]_0^2$

$\quad = 2\left(\frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2} - 0\right) = \frac{32\sqrt{2}}{15}$ //

3.

次の積分を計算せよ．

(1) $\displaystyle\int_1^0 \frac{x}{\sqrt{x}+2}dx$　　(2) $\displaystyle\int e^{\sqrt{x}}dx$　　(3) $\displaystyle\int x^3e^{x^2}dx$

((2) 広島市立大　(3) 信州大)

[解].

(1) $I=\displaystyle\int_1^0 \frac{x}{\sqrt{x}+2}dx$ とおく

$t=\sqrt{x}+2$ とおく　($t=\sqrt{x}$ とおいてもよい)

$\begin{cases} x: 1\to 0 \\ t: 3\to 2 \end{cases}$

$\sqrt{x}=t-2$

$x=t^2-4t+4$, $t\geqq 2$

$\dfrac{dx}{dt}=2t-4$

$I=\displaystyle\int_3^2 \frac{t^2-4t+4}{t}(2t-4)\,dt$

$=-2\displaystyle\int_2^3\left(t^2-6t+12-\frac{8}{t}\right)dt$

$=-2\left[\dfrac{t^3}{3}-3t^2+12t-8\log|t|\right]_2^3$

$=-\dfrac{20}{3}+16\log\dfrac{3}{2}$ //

(2) $I=\displaystyle\int e^{\sqrt{x}}dx$ とおく

$t=\sqrt{x}$ とおく

$x=t^2$, $t\geqq 0$.

$\dfrac{dx}{dt}=2t$

$I=\displaystyle\int e^t\cdot 2t\,dt$

$=2\displaystyle\int te^t dt$　　$f\ e^t \quad g\ t$ / $f'\ e^t \quad g'\ 1$

$=2\left(te^t-\displaystyle\int e^t dt\right)$

$=2(te^t-e^t)$

$=2(t-1)e^t$

$=2(\sqrt{x}-1)e^{\sqrt{x}}+C$ //

(3) $I=\displaystyle\int x^3e^{x^2}dx$

$t=x^2$ とおく

$x=\pm\sqrt{t}$

$\dfrac{dx}{dt}=\pm\dfrac{1}{2\sqrt{t}}$

$I=\displaystyle\int(\pm\sqrt{t})^3\cdot e^t\left(\pm\frac{1}{2\sqrt{t}}\right)dt$

$=\dfrac{1}{2}\displaystyle\int te^t dt$

$=\dfrac{1}{2}(t-1)e^t$

$=\dfrac{1}{2}(x^2-1)e^{x^2}+C$ //

4.

次の積分を計算せよ．

(1) $\displaystyle\int \frac{e^{-2x}}{1+e^{-x}}dx$ (2) $\displaystyle\int_0^1 \frac{1}{2+3e^x+e^{2x}}dx$ (3) $\displaystyle\int \frac{\sin x\cos x}{1+\sin x}dx$

(4) $\displaystyle\int \frac{\cos x}{\sin x(\sin x+1)}dx$ ((1) 関西大 (2) 東京理科大)

[解].

(1) $I=\displaystyle\int \frac{e^{-2x}}{1+e^{-x}}dx$ とおく

$t=e^{-x}$ とおく

$-x=\log t$

$x=-\log t$

$\dfrac{dx}{dt}=-\dfrac{1}{t}$

$I=\displaystyle\int \frac{t^2}{1+t}\cdot\left(-\frac{1}{t}\right)dt$

$=-\displaystyle\int\left(1-\frac{1}{t+1}\right)dt$

$=-(t-\log|t+1|)$

$=-e^{-x}+\log(e^{-x}+1)+C$ //

(2) $I=\displaystyle\int_0^1 \frac{1}{2+3e^x+e^{2x}}dx$

$t=e^x$ とおく $\left(\begin{array}{l} x:0\to 1 \\ t:1\to e \end{array}\right.$

$x=\log t$

$\dfrac{dx}{dt}=\dfrac{1}{t}$

$I=\displaystyle\int_1^e \frac{1}{t^2+3t+2}\cdot\frac{1}{t}dt$

$=\displaystyle\int_1^e \frac{1}{t(t+1)(t+2)}dt$

$\dfrac{1}{t(t+1)(t+2)}=\dfrac{1}{2}\left(\dfrac{1}{t}-\dfrac{2}{t+1}+\dfrac{1}{t+2}\right)$ より

$=\dfrac{1}{2}\Big[\log|t|-2\log|t+1|+\log|t+2|\Big]_1^e$

$=\dfrac{1}{2}\left[\log\dfrac{|t(t+2)|}{(t+1)^2}\right]_1^e$

$=\dfrac{1}{2}\left(\log\dfrac{e(e+2)}{(e+1)^2}-\log\dfrac{3}{4}\right)$

$=\dfrac{1}{2}\log\dfrac{4e(e+2)}{3(e+1)^2}$ //

(3) $I=\displaystyle\int \frac{\sin x\cos x}{1+\sin x}dx$ とおく

$t=\sin x$ とおく

$\dfrac{dx}{dt}=\dfrac{1}{\frac{dt}{dx}}=\dfrac{1}{\cos x}$

$I=\displaystyle\int \frac{\sin x\cancel{\cos x}}{1+\sin x}\cdot\frac{1}{\cancel{\cos x}}dt$

$=\displaystyle\int \frac{t}{1+t}dt$

$=\displaystyle\int\left(1-\frac{1}{t+1}\right)dt$

$=t-\log|t+1|$

$=\sin x-\log(\sin x+1)+C$ //

(4) $I=\displaystyle\int \frac{\cos x}{\sin x(\sin x+1)}dx$ とおく

$t=\sin x$ とおく

$\dfrac{dx}{dt}=\dfrac{1}{\frac{dt}{dx}}=\dfrac{1}{\cos x}$

$I=\displaystyle\int \frac{\cancel{\cos x}}{\sin x(\sin x+1)}\cdot\frac{1}{\cancel{\cos x}}dt$

$=\displaystyle\int \frac{1}{t(t+1)}dt$

$=\displaystyle\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$

$=\log|t|-\log|t+1|$

$=\log\dfrac{|t|}{|t+1|}$

$=\log\dfrac{|\sin x|}{\sin x+1}+C$ //

5.

次の定積分を求めよ.

(1) $\displaystyle\int_0^a \sqrt{a^2-x^2}dx\ (a>0)$　　(2) $\displaystyle\int_0^1 (1+x)\sqrt{1-x^2}dx$

(3) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

[解]

(1) $I=\int_0^a\sqrt{a^2-x^2}\,dx$ とおく

$x=a\sin\theta$ とおく

$\begin{cases} x:0\to a \\ \theta:0\to\frac{\pi}{2}\end{cases}$

$\frac{dx}{d\theta}=a\cos\theta$

$I=\int_0^{\frac{\pi}{2}}\sqrt{a^2(1-\sin^2\theta)}\cdot a\cos\theta\,d\theta$

$=a^2\int_0^{\frac{\pi}{2}}|\cos\theta|\cos\theta\,d\theta$

$=a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$ ← $0\le\theta\le\frac{\pi}{2}$ で $\cos\theta\ge0$

$=a^2\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta}{2}\,d\theta$

$=\frac{a^2}{2}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$=\frac{\pi}{4}a^2$ //

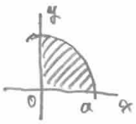
[別解]

$y=\sqrt{a^2-x^2}$ は

$y^2=a^2-x^2,\ y\ge0$

$x^2+y^2=a^2$ (円) の $y\ge0$ (上半分) の部分.

I は $0\le x\le a$ において $y=\sqrt{a^2-x^2}$ と x 軸とで囲まれる面積より



$I=\frac{1}{2}a^2\cdot\frac{\pi}{2}=\frac{\pi}{4}a^2$ //

(2) $I=\int_0^1(1+x)\sqrt{1-x^2}\,dx$ とおく

$=\int_0^1\sqrt{1-x^2}\,dx-\frac{1}{2}\int_0^1\sqrt{1-x^2}\cdot(1-x^2)'\,dx$

$=\frac{1}{2}\cdot1^2\cdot\frac{\pi}{2}-\frac{1}{2}\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$=\frac{\pi}{4}-\frac{1}{3}(0-1)$

$=\frac{\pi}{4}+\frac{1}{3}$ //

(3) $I=\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}$ とおく

$x=\sin\theta$ とおく

$\begin{cases} x:0\to\frac{1}{2} \\ \theta:0\to\frac{\pi}{6}\end{cases}$

$\frac{dx}{d\theta}=\cos\theta$

$I=\int_0^{\frac{\pi}{6}}\frac{1}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta$

$=\int_0^{\frac{\pi}{6}}\frac{1}{|\cos\theta|}\cos\theta\,d\theta$

$=\int_0^{\frac{\pi}{6}}\frac{1}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta$

$=[\theta]_0^{\frac{\pi}{6}}=\frac{\pi}{6}$ //

〈point〉

① $\sqrt{a^2-x^2}$ を含む積分

$x=a\sin\theta$ とおく.

6.

次の定積分を求めよ．

(1) $\displaystyle\int_0^1 \frac{dx}{1+x^2}$　　(2) $\displaystyle\int_0^1 \frac{dx}{(3+x^2)^2}$

(3) $\displaystyle\int_0^1 \log(1+x^2)dx$

((2) 弘前大　(3) 福島県立医大)

[解]

(1) $I=\displaystyle\int_0^1 \frac{dx}{1+x^2}$ とおく

$x=\tan\theta$ とおく

$\begin{cases} x: 0\to 1 \\ \theta: 0\to \frac{\pi}{4} \end{cases}$

$\dfrac{dx}{d\theta}=\dfrac{1}{\cos^2\theta}=1+\tan^2\theta$

$I=\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\cancel{1+\tan^2\theta}}\cdot\cancel{(1+\tan^2\theta)}\,d\theta$

$=[\theta]_0^{\frac{\pi}{4}}=\dfrac{\pi}{4}$ //

(2) $I=\displaystyle\int_0^1 \frac{dx}{(3+x^2)^2}$ とおく．

$x=\sqrt{3}\tan\theta$ とおく

$\begin{cases} x: 0\to 1 \\ \theta: 0\to \frac{\pi}{6} \end{cases}$

$\dfrac{dx}{d\theta}=\sqrt{3}(1+\tan^2\theta)$

$I=\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{9(1+\tan^2\theta)^{\cancel{2}}}\cdot\sqrt{3}\cancel{(1+\tan^2\theta)}\,d\theta$

$=\dfrac{\sqrt{3}}{9}\displaystyle\int_0^{\frac{\pi}{6}}\cos^2\theta\,d\theta$

$=\dfrac{\sqrt{3}}{9}\displaystyle\int_0^{\frac{\pi}{6}}\frac{1+\cos 2\theta}{2}\,d\theta$

$=\dfrac{\sqrt{3}}{18}\left[\theta+\dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$

$=\dfrac{\sqrt{3}}{18}\left(\dfrac{\pi}{6}+\dfrac{\sqrt{3}}{4}\right)$ //

(3) $I=\displaystyle\int_0^1 \log(1+x^2)\,dx$ とおく　f x　f' 1　g $\log(1+x^2)$　g' $\dfrac{2x}{1+x^2}$

$=\left[x\log(1+x^2)\right]_0^1-2\displaystyle\int_0^1\frac{x^2}{1+x^2}\,dx$

$=\log 2-2\displaystyle\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$

ここで

$\displaystyle\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}$ ← 左の結果より

より

$=\log 2-2+\dfrac{\pi}{2}$ //

〈point〉

① 分母に a^2+x^2 を含む積分

$x=a\tan\theta$ とおく．

1.

次の定積分を求めよ.

(1) $\displaystyle\int_0^3 \sqrt{6x-x^2}\,dx$　　　(2) $\displaystyle\int_1^3 \frac{dx}{\sqrt{4x-x^2}}$

[解]

(1) $I=\int_0^3\sqrt{6x-x^2}\,dx$ とおく

$y=\sqrt{6x-x^2}$ は

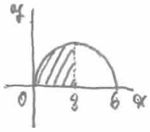
$y^2=6x-x^2$, $y\geqq 0$

$x^2-6x+y^2=0$

$(x-3)^2+y^2=9$ (円) の $y\geqq 0$ (上半分) の

部分.

$I=\frac{1}{2}\cdot 3^2\cdot\frac{\pi}{2}=\frac{9}{4}\pi$ //



(2) $I=\int_1^3\frac{dx}{\sqrt{4x-x^2}}$ とおく

$=\int_1^3\frac{dx}{\sqrt{4-(x-2)^2}}$

$x-2=2\sin\theta$ とおく

$\begin{pmatrix} x: 1\to 3 \\ \theta: -\frac{\pi}{6}\to\frac{\pi}{6}\end{pmatrix}$

$x=2\sin\theta+2$

$\frac{dx}{d\theta}=2\cos\theta$

$I=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta$

$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{\cancel{2\cos\theta}}\cancel{2\cos\theta}\,d\theta$ ← $-\frac{\pi}{6}\leqq\theta\leqq\frac{\pi}{6}$ で $\cos\theta\geqq 0$

$=[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=\frac{\pi}{3}$ //

2.

次の定積分を求めよ．

(1) $\displaystyle\int_{-1}^{0} \frac{1}{x^2+2x+2}dx$　　　(2) $\displaystyle\int_{0}^{1} \frac{1}{x^3+1}dx$

[解]

(1) $I = \displaystyle\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx$ とおく

$= \displaystyle\int_{-1}^{0} \frac{1}{1+(x+1)^2}\,dx$

$x+1 = \tan\theta$ とおく

$\begin{cases} x: -1 \to 0 \\ \theta: 0 \to \frac{\pi}{4} \end{cases}$

$x = \tan\theta - 1$

$\dfrac{dx}{d\theta} = 1 + \tan^2\theta$

$I = \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\cancel{1+\tan^2\theta}} \cdot \cancel{(1+\tan^2\theta)}\,d\theta$

$= [\theta]_0^{\frac{\pi}{4}} = \dfrac{\pi}{4}$ //

(2) $I = \displaystyle\int_0^1 \frac{1}{x^3+1}\,dx$ とおく

$\dfrac{1}{x^3+1} = \dfrac{a}{x+1} + \dfrac{bx+c}{x^2-x+1}$ とおく

$= \dfrac{a(x^2-x+1)+(bx+c)(x+1)}{x^3+1}$

$= \dfrac{(a+b)x^2+(-a+b+c)x+a+c}{x^3+1}$

$\begin{cases} a+b=0 \\ -a+b+c=0 \\ a+c=1 \end{cases}$　　$\therefore a = \frac{1}{3},\ b = -\frac{1}{3},\ c = \frac{2}{3}$

$\therefore \dfrac{1}{x^3+1} = \dfrac{1}{3}\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)$

$I = \dfrac{1}{3}\displaystyle\int_0^1 \Big(\underbrace{\frac{1}{x+1}}_{I_1} - \underbrace{\frac{x-2}{x^2-x+1}}_{I_2}\Big)\,dx$ とおく

$I_1 = \displaystyle\int_0^1 \frac{1}{x+1}\,dx$

$= [\log|x+1|]_0^1 = \log 2$

$I_2 = \displaystyle\int_0^1 \frac{x-2}{x^2-x+1}\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^1 \Big(\underbrace{\frac{2x-1}{x^2-x+1}}_{I_3} - \underbrace{\frac{3}{x^2-x+1}}_{I_4}\Big)\,dx$ とおく

$I_3 = \displaystyle\int_0^1 \frac{(x^2-x+1)'}{x^2-x+1}\,dx$

$= [\log(x^2-x+1)]_0^1 = 0$

$I_4 = \displaystyle\int_0^1 \frac{3}{x^2-x+1}\,dx$

$= 3\displaystyle\int_0^1 \frac{1}{\frac{3}{4}+(x-\frac{1}{2})^2}\,dx$

$x - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\tan\theta$ とおく

$\begin{cases} x: 0 \to 1 \\ \theta: -\frac{\pi}{6} \to \frac{\pi}{6} \end{cases}$

$x = \dfrac{\sqrt{3}}{2}\tan\theta + \dfrac{1}{2}$

$\dfrac{dx}{d\theta} = \dfrac{\sqrt{3}}{2}(1+\tan^2\theta)$

$I_4 = 2\sqrt{3}\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{\cancel{1+\tan^2\theta}} \cdot \cancel{(1+\tan^2\theta)}\,d\theta$

$= 2\sqrt{3}\,[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \dfrac{2\sqrt{3}}{3}\pi$

よって

$I_2 = -\dfrac{\sqrt{3}}{3}\pi$

$\therefore I = \dfrac{1}{3}\left(\log 2 + \dfrac{\sqrt{3}}{3}\pi\right)$ //

3.

(1) 次の式が成り立つように，定数 A，B，C，D を定めよ．

$$\frac{8}{x^4+4}=\frac{Ax+B}{x^2+2x+2}+\frac{Cx+D}{x^2-2x+2}$$

(2) $\tan\frac{\pi}{8}$，$\tan\frac{3}{8}\pi$ の値を求めよ．

(3) 次の定積分を求めよ．

$$\int_{-\sqrt{2}}^{\sqrt{2}}\frac{8}{x^2+4}dx$$

（信州大）

[解]

(1)

$$\frac{8}{x^4+4}=\frac{Ax+B}{x^2+2x+2}+\frac{Cx+D}{x^2-2x+2}$$

$$=\frac{(Ax+B)(x^2-2x+2)+(Cx+D)(x^2+2x+2)}{x^4+4}$$

$$=\frac{(A+C)x^3+(-2A+B+2C+D)x^2+(2A-2B+2C+2D)x+2B+2D}{x^4+4}$$

$$\begin{cases} A+C=0 \\ -2A+B+2C+D=0 \\ A-B+C+D=0 \\ B+D=4 \end{cases}$$

よって

$A=1\quad B=2\quad C=-1\quad D=2$

(2)

$$\tan^2\frac{\pi}{8}=\frac{\sin^2\frac{\pi}{8}}{\cos^2\frac{\pi}{8}}=\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\frac{2-\sqrt{2}}{2+\sqrt{2}}=3-2\sqrt{2}$$

$\tan\frac{\pi}{8}>0$ より

$$\tan\frac{\pi}{8}=\sqrt{3-2\sqrt{2}}=\sqrt{2}-1$$

$$\tan^2\frac{3}{8}\pi=\frac{1-\cos\frac{3}{4}\pi}{1+\cos\frac{3}{4}\pi}=\frac{2+\sqrt{2}}{2-\sqrt{2}}=3+2\sqrt{2}$$

$\tan\frac{3}{8}\pi>0$ より

$$\tan\frac{3}{8}\pi=\sqrt{2}+1$$

(3) $I=\int_{-\sqrt{2}}^{\sqrt{2}}\frac{8}{x^2+4}dx$ とおく

$$=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}\right)dx$$

$$=\underbrace{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\frac{2x+2}{x^2+2x+2}dx}_{I_1}+\underbrace{\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{x^2+2x+2}dx}_{I_2}$$

$$-\underbrace{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\frac{2x-2}{x^2-2x+2}dx}_{I_3}+\underbrace{\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{x^2-2x+2}dx}_{I_4}$$ とおく

$$I_1=\left[\log(x^2+2x+2)\right]_{-\sqrt{2}}^{\sqrt{2}}=\log(4+2\sqrt{2})-\log(4-2\sqrt{2})=\log\frac{4+2\sqrt{2}}{4-2\sqrt{2}}$$

$$I_3=\left[\log(x^2-2x+2)\right]_{-\sqrt{2}}^{\sqrt{2}}=\log(4-2\sqrt{2})-\log(4+2\sqrt{2})=\log\frac{4-2\sqrt{2}}{4+2\sqrt{2}}$$

$$I_1-I_3=2\log\frac{4+2\sqrt{2}}{4-2\sqrt{2}}=2\log(3+2\sqrt{2})$$

$$I_2=\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{x^2+2x+2}dx=\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{(x+1)^2+1}dx$$

$x+1=\tan\theta$ とおくと

$$\begin{cases} x:-\sqrt{2}\to\sqrt{2} \\ \theta:-\frac{\pi}{8}\to\frac{3}{8}\pi \end{cases}$$ （(2)より）

$$\frac{dx}{d\theta}=1+\tan^2\theta$$

$$I_2=\int_{-\frac{\pi}{8}}^{\frac{3}{8}\pi}\frac{1}{\cancel{\tan^2\theta+1}}\cancel{(1+\tan^2\theta)}\,d\theta$$

$$=\left[\theta\right]_{-\frac{\pi}{8}}^{\frac{3}{8}\pi}=\frac{\pi}{2}$$

$$I_4 = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{x^2-2x+2}\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{(x-1)^2+1}\,dx$$

$x-1=\tan\theta$ とおくと

$$\begin{cases} x : -\sqrt{2} \to \sqrt{2} \\ \theta : -\frac{3}{8}\pi \to \frac{\pi}{8} \end{cases}$$

同様にして

$$I_4 = \frac{\pi}{2}$$

よって

$$I = \log(3+2\sqrt{2}) + \pi$$