$I=\int_0^{\frac{\pi}{2}}\left(\frac{x\sin x}{1+\cos x}+\frac{x\cos x}{1+\sin x}\right)dx$ とおく．

(1)　等式　$I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{1+\cos x}+\frac{\cos x}{1+\sin x}\right)dx$ を示せ．

(2)　I の値を求めよ．

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解説

(1) $x=\frac{\pi}{2}-t$ とおくと，$\frac{dx}{dt}=-1$

$$I=\int_{\frac{\pi}{2}}^{0}\left(\frac{\pi}{2}-t\right)\left(\frac{\cos t}{1+\sin t}+\frac{\sin t}{1+\cos t}\right)\cdot(-1)dt$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{\sin t}{1+\cos t}+\frac{\cos t}{1+\sin t}\right)dt-I$$

$$\therefore\ I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{1+\cos x}+\frac{\cos x}{1+\sin x}\right)dx$$

(2) (1)より

$$I=\frac{\pi}{4}\Big[-\log(1+\cos x)+\log(1+\sin x)\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4}(\log 2+\log 2)=\frac{\pi}{2}\log 2$$