$f(x)$ が $0 \leqq x \leqq 1$ で連続な関数であるとき

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

が成立することを示し，これを用いて定積分 $\displaystyle\int_0^\pi \frac{x\sin x}{3+\sin^2 x}dx$ を求めよ．

(99　信州大)

解説

$I = \displaystyle\int_0^\pi xf(\sin x)\,dx$ とおく

$x = \pi - t$ とおくと，$\dfrac{dx}{dt} = -1$

$$I = \int_\pi^0 (\pi - t)f(\sin(\pi - t))\cdot(-1)dt$$

$$= \int_0^\pi (\pi - t)f(\sin t)dt$$

$$= \pi\int_0^\pi f(\sin t)dt - I$$

$$2I = \pi\int_0^\pi f(\sin x)dx$$

$$\therefore\ I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

$f(x) = \dfrac{x}{3+x^2}$ とおくと，$f(x)$ は $0 \leqq x \leqq 1$ で連続な関数であるから

$$\int_0^\pi \frac{x\sin x}{3+\sin^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{3+\sin^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{4-\cos^2 x}dx$$

$\cos x = t$ とおくと，$\dfrac{dx}{dt} = \dfrac{1}{\dfrac{dt}{dx}} = -\dfrac{1}{\sin x}$

$$= \frac{\pi}{2}\int_1^{-1} \frac{1}{4-t^2}\cdot(-1)dt = \frac{\pi}{2}\int_{-1}^1 \frac{dt}{4-t^2} = \pi\int_0^1 \frac{dt}{4-t^2}$$

$$= \frac{\pi}{4}\int_0^1\left(\frac{1}{2+t} + \frac{1}{2-t}\right)dt = \frac{\pi}{4}\left[\log\left(\frac{2+t}{2-t}\right)\right]_0^1 = \frac{\pi}{4}\log 3$$