発展問題演習 10. 体積(1)

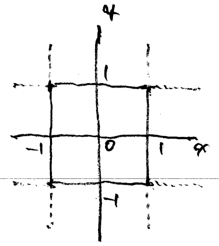
[1]

立体Pの体積を$V(P)$と表す

$V(A)=V(B)=V(C)=\pi\cdot1^2\cdot2=2\pi$

$V(A\cap B)$

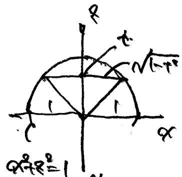
立体$A\cap B$を$z=t\ (-1\leqq t\leqq 1)$で切ると断面積$S(t)$は

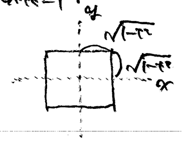


1つ柱が交差している

$$S(t)=(2\sqrt{1-t^2})^2$$
$$=4(1-t^2)$$

より

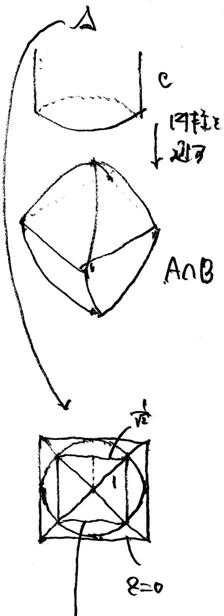




$$V(A\cap B)=\int_{-1}^{1}S(t)\,dt$$
$$=4\int_{-1}^{1}(1-t^2)\,dt$$
$$=8\int_{0}^{1}(1-t^2)\,dt$$
$$=8\left[t-\frac{t^3}{3}\right]_0^1=\frac{16}{3}$$

同様にして

$$V(B\cap C)=V(C\cap A)=\frac{16}{3}$$



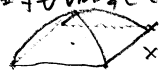
$\sqrt{1-t^2}=\frac{1}{\sqrt{2}}$

$1-t^2=\frac{1}{2}$ ∴$t=\pm\frac{1}{\sqrt{2}}$

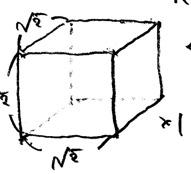
$z=\pm\frac{1}{\sqrt{2}}$

$V(A\cap B\cap C)$

立体$A\cap B\cap C$は
対称性も加味して



各面にくっついている



よって

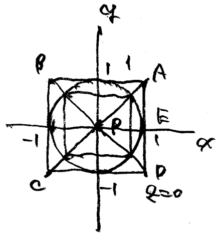
$$V(A\cap B\cap C)=(\sqrt{2})^3+6\int_{\frac{1}{\sqrt{2}}}^{1}4(1-t^2)\,dt$$
$$=2\sqrt{2}+24\left[t-\frac{t^3}{3}\right]_{\frac{1}{\sqrt{2}}}^{1}$$
$$=2\sqrt{2}+24\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)$$
$$=16-8\sqrt{2}$$

したがって

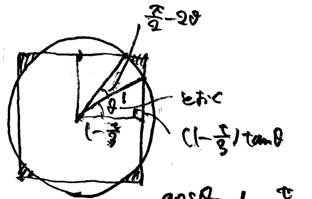
$$V(A\cup B\cup C)=V(A)+V(B)+V(C)$$
$$-V(A\cap B)-V(B\cap C)-V(C\cap A)$$
$$+V(A\cap B\cap C)$$
$$=6\pi-3\cdot\frac{16}{3}+16-8\sqrt{2}$$
$$=6\pi-8\sqrt{2}$$

[2]

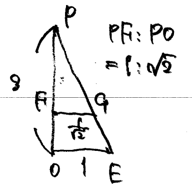
$z=t\ \left(0\leqq t\leqq 3\left(1-\frac{1}{\sqrt{2}}\right)\right)$で切ると



上から見た図



$\cos\theta=1-\frac{t}{3}$

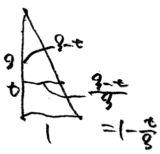


PF : PO $=1:\sqrt{2}$

$OF=3\cdot\frac{\sqrt{2}-1}{\sqrt{2}}$
$=3\left(1-\frac{1}{\sqrt{2}}\right)$

断面積$S(t)$は ∵$t=3(1-\cos\theta)$

$$S(t)=4\left\{\left(1-\frac{t}{3}\right)^2-\frac{1}{2}\left(1-\frac{t}{3}\right)^2\tan\theta-\frac{1}{2}\cdot1^2\left(\frac{\pi}{2}-2\theta\right)\right\}$$
$$=4\left\{\cos^2\theta-\frac{1}{2}\sin2\theta+\theta-\frac{\pi}{4}\right\}$$
$$=4\cos^2\theta-2\sin2\theta+4\theta-\pi$$



求める体積Vは

$$V=\int_0^{3(1-\frac{1}{\sqrt{2}})}S(t)\,dt$$

$t=3(1-\cos\theta)$ より $\frac{dt}{d\theta}=3\sin\theta$　　$t:0\to3\left(1-\frac{1}{\sqrt{2}}\right)$
$\theta:0\to\frac{\pi}{4}$

$$=\int_0^{\frac{\pi}{4}}(4\cos^2\theta-2\sin2\theta+4\theta-\pi)\cdot3\sin\theta\,d\theta$$

ここで

$$\int_0^{\frac{\pi}{4}}\cos^2\theta\sin\theta\,d\theta=-\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{4}}=\frac{1}{3}\left(1-\frac{\sqrt{2}}{4}\right)$$

$$\int_0^{\frac{\pi}{4}}\underbrace{\sin2\theta}_{2\sin\theta\cos\theta}\sin\theta\,d\theta=2\left[\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{4}}=\frac{2}{3}\cdot\frac{\sqrt{2}}{4}=\frac{\sqrt{2}}{6}$$

$$\int_0^{\frac{\pi}{4}}\theta\sin\theta\,d\theta=\left[-\theta\cos\theta\right]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\cos\theta\,d\theta$$　$f\ \theta$　$g\ -\cos\theta$，$f'\ 1$　$g'\ \sin\theta$

$$=-\frac{\sqrt{2}}{8}\pi+\left[\sin\theta\right]_0^{\frac{\pi}{4}}=-\frac{\sqrt{2}}{8}\pi+\frac{\sqrt{2}}{2}$$

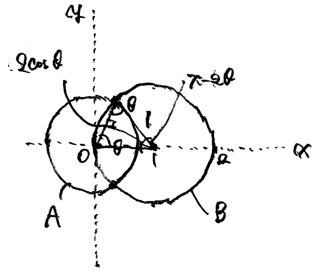
$$\int_0^{\frac{\pi}{4}}\sin\theta\,d\theta=\left[-\cos\theta\right]_0^{\frac{\pi}{4}}=1-\frac{\sqrt{2}}{2}$$

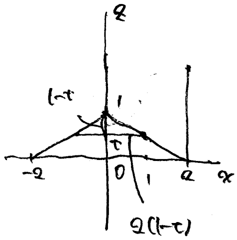
より

$$V=4\left(1-\frac{\sqrt{2}}{4}\right)-\sqrt{2}+12\left(-\frac{\sqrt{2}}{8}\pi+\frac{\sqrt{2}}{2}\right)-3\pi\left(1-\frac{\sqrt{2}}{2}\right)$$
$$=4-\sqrt{2}-\sqrt{2}-\cancel{\frac{3\sqrt{2}}{2}\pi}+6\sqrt{2}-3\pi+\cancel{\frac{3\sqrt{2}}{2}\pi}$$
$$=4+4\sqrt{2}-3\pi$$

$\boxed{3}$

(1) $z=t\ (0\leqq t\leqq 1)$ で切ると





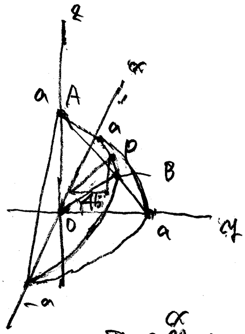
$t=1-\cos\theta$ とすると
$\cos\theta=1-t$

$$S(t)=\frac{1}{2}\cdot 1^2\cdot 2(\pi-2\theta)-\frac{1}{2}\cdot 1\cdot 1\cdot\sin(2\pi-4\theta)$$
$$+\frac{1}{2}\cdot 4\cos^2\theta\cdot 2\theta-\frac{1}{2}\cdot 4\cos^2\theta\sin 2\theta$$
$$=\pi-2\theta+\frac{1}{2}\underbrace{\sin 4\theta}_{\sin 2\theta\cos 2\theta}+4\theta\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}-2\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\sin 2\theta$$
$$=\pi+2\theta\cos 2\theta+\sin 2\theta(\cos 2\theta-2\cos^2\theta)$$
$$=\pi+2\theta\cos 2\theta-\sin 2\theta$$

(2) 求める体積Vは
$$V=\int_0^1 S(t)\,dt$$
$t=1-\cos\theta$ より $\dfrac{dt}{d\theta}=\sin\theta$　　$t: 0\to 1$, $\theta: 0\to\dfrac{\pi}{2}$
$$V=\int_0^{\frac{\pi}{2}}(\pi+2\theta\cos 2\theta-\sin 2\theta)\sin\theta\,d\theta$$
ここで
$$\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta=\left[-\cos\theta\right]_0^{\frac{\pi}{2}}=1$$
$s(2\theta+\theta)=s2\theta c\theta+c2\theta s\theta$
$-)\ s(2\theta-\theta)=s2\theta c\theta-c2\theta s\theta$
$$\int_0^{\frac{\pi}{2}}2\theta\cos 2\theta\sin\theta\,d\theta=\int_0^{\frac{\pi}{2}}\theta(\sin 3\theta-\sin\theta)\,d\theta$$
$$=-\frac{10}{9}$$
$$\int_0^{\frac{\pi}{2}}\sin 2\theta\cos\theta=\int_0^{\frac{\pi}{2}}2\cos^2\theta\cdot\sin\theta\,d\theta=\frac{2}{3}$$
より
$$V=\pi-\frac{10}{9}-\frac{2}{3}=\pi-\frac{16}{9}$$

$\boxed{4}$

(1) 切り口の境界の曲線 C 上の点を $P(x,y,z)$ とする
Pのy座標とz座標は等しいから
$P(x,y,y)$ とおける
Pは円すいの側面上の点より
$\vec{AP}$ と $\vec{AO}$ のなす角は45°より



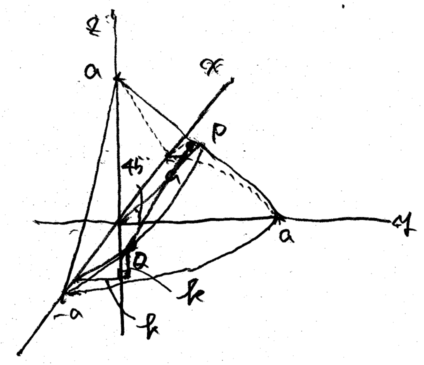
$\vec{AP}=\begin{pmatrix}x\\y\\y-a\end{pmatrix}$
$\vec{AO}=\begin{pmatrix}0\\0\\-1\end{pmatrix}$

$$\vec{AP}\cdot\vec{AO}=|\vec{AP}||\vec{AO}|\cos 45^\circ$$
$$a-y=\sqrt{x^2+y^2+(y-a)^2}\cdot\frac{1}{\sqrt{2}}$$
$$\sqrt{2}(a-y)=\sqrt{x^2+2y^2-2ay+a^2}$$
$$2(a-y)^2=x^2+\cancel{2y^2}-2ay+a^2,\quad y\leqq a$$
$$a^2-2ay+\cancel{y^2}$$
$$2ay=-x^2+a^2$$
$$\therefore\ y=-\frac{1}{2a}(x^2-a^2)$$

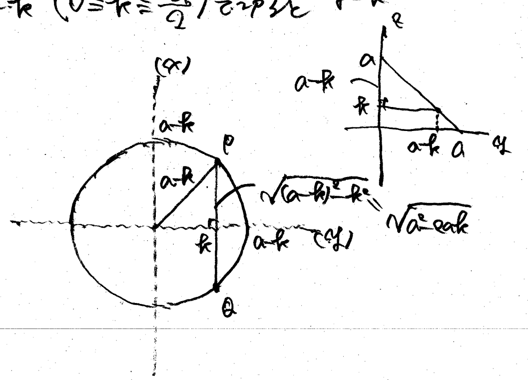
よって曲線Cのxy平面上の正射影の方程式は
$$y=-\frac{1}{2a}(x^2-a^2)$$
この正射影とx軸とで囲まれる部分の面積をS'とすると
切り口の面積Sは
$$S\cos 45^\circ=S'$$
$$S=\sqrt{2}S'=\sqrt{2}\int_{-a}^{a}-\frac{1}{2a}(x^2-a^2)\,dx$$
$$=-\frac{\sqrt{2}}{2a}\int_{-a}^{a}(x+a)(x-a)\,dx$$
$$=\frac{\sqrt{2}}{12a}(2a)^3=\frac{2\sqrt{2}}{3}a^2$$

(2) もとの円すいの体積は
$$\frac{1}{3}\pi a^2 a=\frac{\pi}{3}a^3$$
2つの部分に分けたうち小さい方の体積は
切り口⊥ABより
$$\frac{1}{3}\cdot\frac{2\sqrt{2}}{3}a^2\cdot\frac{a}{\sqrt{2}}=\frac{2}{9}a^3$$
よって求める体積は
$$\frac{1}{2}\cdot\frac{\pi}{3}a^3-\frac{2}{9}a^3=\frac{3\pi-4}{18}a^3$$

4 川口君の解答の研究



$z=k$ $\left(0\leqq k\leqq \frac{a}{2}\right)$ で切ると $y=k$



切り口を xy 平面に正射影したものの面積 S' は

$$S'=\int_0^{\frac{a}{2}} 2\sqrt{a^2-2ak}\,dk$$

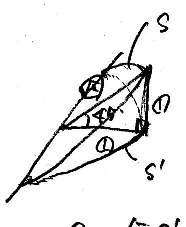
$$=2\left[\frac{(a^2-2ak)^{\frac{3}{2}}}{\frac{3}{2}\cdot(-2a)}\right]_0^{\frac{a}{2}}$$

$$=-\frac{2}{3a}\left[0-a^3\right]=\frac{2}{3}a^2$$

切り口の面積を S とすると

$$S=\sqrt{2}\,S'$$

$$=\frac{2\sqrt{2}}{3}a^2$$



$S=\sqrt{2}\,S'$

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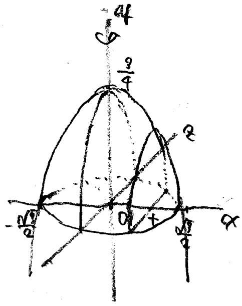
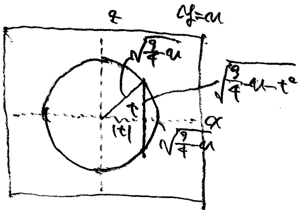
xy平面において $y=u\ (u\leqq\frac{9}{4})$ のとき

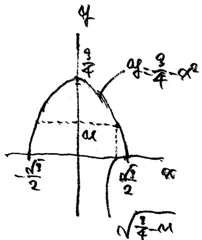
$u=\frac{9}{4}-x^2$

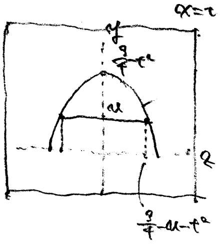
$x^2=\frac{9}{4}-u\quad \therefore x=\pm\sqrt{\frac{9}{4}-u}$

平面Kを $x=t$ で切ると









$$\begin{cases} y=u \cdots ① \\ z=\pm\sqrt{\frac{9}{4}-u-t^2} \cdots ② \end{cases}$$

②より

$z^2=\frac{9}{4}-u-t^2$

$\therefore u=-z^2+\frac{9}{4}-t^2$

切り口の曲線の方程式は

$y=-z^2+\frac{9}{4}-t^2$

KとHで囲まれた立体を $x=t$ で切ると

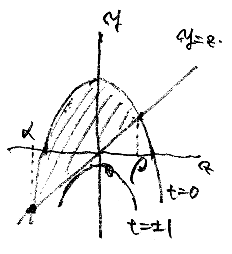
(平面Hを $y=z$ となるように設定する)

断面積 $S(t)$ は

$-z^2+\frac{9}{4}-t^2=z$ として

$z^2+z+t^2-\frac{9}{4}=0$

このz解を$\alpha,\beta\ (\alpha<\beta)$として



$$S(t)=\int_\alpha^\beta\left(-z^2+\frac{9}{4}-t^2-z\right)dz$$

$$=-\int_\alpha^\beta(z-\alpha)(z-\beta)dz$$

$$=\frac{1}{6}(\beta-\alpha)^3$$

$$=\frac{1}{6}\left(2\sqrt{1-t^2}\right)^3$$

$$=\frac{4}{3}\left(\sqrt{1-t^2}\right)^3 \quad (-1\leqq t\leqq 1)$$

$$z=\frac{-1\pm\sqrt{1-4(t^2-\frac{9}{4})}}{2}=\frac{-1\pm2\sqrt{1-t^2}}{2}$$

求める体積Vは

$$V=\int_{-1}^{1}\frac{4}{3}\left(\sqrt{1-t^2}\right)^3dt$$

$$=\frac{8}{3}\int_0^1\left(\sqrt{1-t^2}\right)^3dt$$

$t=\sin\theta$ とおくと $\frac{dt}{d\theta}=\cos\theta$ 　　$t:0\to1$, $\theta:0\to\frac{\pi}{2}$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\cdot\cos\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\, d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos2\theta}{2}\right)^2d\theta$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}(1+2\cos2\theta+\underbrace{\cos^2 2\theta}_{\frac{1+\cos4\theta}{2}})d\theta$$

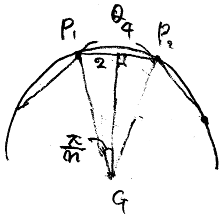
$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\left(\frac{3}{2}+2\cos2\theta+\frac{1}{2}\cos4\theta\right)d\theta$$

$$=\frac{2}{3}\left[\frac{3}{2}\theta+\sin2\theta+\frac{1}{8}\sin4\theta\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$ //

7

(1) $GQ\tan\frac{\pi}{n}=2$ より



$GQ=\frac{2}{\tan\frac{\pi}{n}}$

$n\geqq3$ であるから $0<\frac{\pi}{n}\leqq\frac{\pi}{3}$

$0<\theta<\frac{\pi}{2}$ で $\tan\theta$ は単調増加であるから

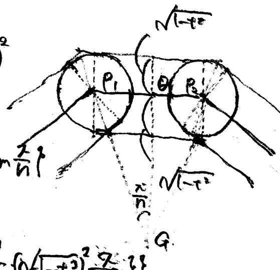
$\tan\frac{\pi}{n}\leqq\sqrt{3}$ より

$GQ\geqq\frac{2}{\sqrt{3}}>1$ //

$GP_1\sin\frac{\pi}{n}=2$

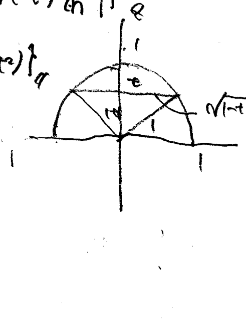
(2) $S(t)=n\Big\{2\cdot\frac{1}{2}\left(\frac{2}{\tan\frac{\pi}{n}}+\sqrt{1-t^2}\right)^2\tan\frac{\pi}{n}$

$-2\cdot\frac{1}{2}\left(\frac{2}{\tan\frac{\pi}{n}}-\sqrt{1-t^2}\right)^2\tan\frac{\pi}{n}\Big\}$



$-2\left\{\frac{1}{2}\left(\sqrt{1-t^2}\right)^2\tan\frac{\pi}{n}-\frac{1}{2}\left(\sqrt{1-t^2}\right)^2\frac{\pi}{n}\right\}\Big\}$

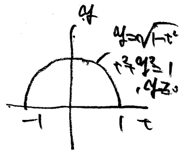
$=n\left\{8\sqrt{1-t^2}+\left(\tan\frac{\pi}{n}-\frac{\pi}{n}\right)(1-t^2)\right\}$ //



(2) $V(n)=\int_{-1}^{1} S(t)\,dt$

ここで

$$\int_{-1}^{1}\sqrt{1-t^2}\,dt=\frac{1}{2}\cdot 1^2\cdot\pi=\frac{\pi}{2}$$



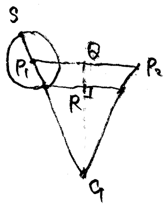
$$\int_{-1}^{1}(1-t^2)\,dt=2\left[t-\frac{t^3}{3}\right]_0^1=\frac{4}{3}$$

より

$$V(n)=4n\pi-\frac{4}{3}n\tan\frac{\pi}{n}+\frac{4}{3}\pi$$

(3) この立体を $z=t$ で切ったとき

断面積 $S'(t)$ は



$$S'(t)=\pi(GS^2-GR^2)$$

$$=\pi\left\{\left(\frac{2}{\sin\frac{\pi}{n}}+\sqrt{1-t^2}\right)^2-\left(\frac{2}{\tan\frac{\pi}{n}}-\sqrt{1-t^2}\right)^2\right\}$$

$\frac{1}{\sin^2\frac{\pi}{n}}=1+\frac{1}{\tan^2\frac{\pi}{n}}$

$$=\pi\left\{\frac{4}{\sin^2\frac{\pi}{n}}-\frac{4}{\tan^2\frac{\pi}{n}}+4\left(\frac{1}{\sin\frac{\pi}{n}}+\frac{1}{\tan\frac{\pi}{n}}\right)\sqrt{1-t^2}\right\}$$

$$=4\pi+4\pi\cdot\frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}\sqrt{1-t^2}$$

$$W(n)=\int_{-1}^{1} S'(t)\,dt$$

$$=8\pi+2\pi^2\cdot\frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$ //

(4)

$$\lim_{n\to\infty}\frac{V(n)}{W(n)}=\lim_{n\to\infty}\frac{4n\pi-\frac{4}{3}n\tan\frac{\pi}{n}+\frac{4}{3}\pi}{8\pi+2\pi^2\cdot\frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}}$$

ここで、$t=\frac{\pi}{n}$ とおくと $\left(n=\frac{\pi}{t}\right)$

$n\to\infty$ のとき $t\to 0$ より

$$=\lim_{t\to 0}\frac{\frac{4\pi^2}{t}-\frac{4}{3}\pi\frac{\tan t}{t}+\frac{4}{3}\pi}{8\pi+2\pi^2\frac{1+\cos t}{\sin t}}$$

$$=\lim_{t\to 0}\frac{4\pi^2-\frac{4}{3}\pi\tan t+\frac{4}{3}\pi t}{8\pi t+2\pi^2\cdot\frac{t}{\sin t}(1+\cos t)}$$

$$=\frac{4\pi^2}{4\pi^2}=1.$$ //

$\boxed{8}$

(1) $z=t\ (0\leqq t\leqq 1)$ で切ると

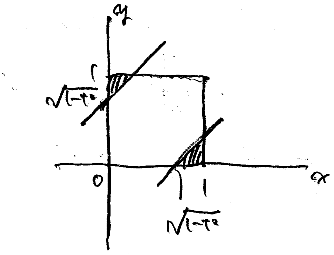
$0\leqq x\leqq 1,\ 0\leqq y\leqq 1$

$x^2+y^2+t^2-2xy-1\geqq 0$

$(x-y)^2\geqq 1-t^2 \quad (1-t^2\geqq 0)$

$x-y\geqq\sqrt{1-t^2},\ x-y\leqq-\sqrt{1-t^2}$

$\therefore\ y\leqq x-\sqrt{1-t^2},\ y\geqq x+\sqrt{1-t^2}$



よって

$$S(t)=2\cdot\frac{1}{2}\left(1-\sqrt{1-t^2}\right)^2$$

$$=2-t^2-2\sqrt{1-t^2}$$ //

(2) 求める体積 V は

$$V=\int_0^1 S(t)\,dt$$

$$=\left[2t-\frac{t^3}{3}\right]_0^1-2\cdot\frac{1}{2}\cdot 1^2\cdot\frac{\pi}{2}$$

$$=\frac{5}{3}-\frac{\pi}{2}$$ //