

89 [茨城大]

自然数  $n$  に対して、 $a_n = 2^n + 3^n + 1$  とおくとき、

- (1)  $a_{n+6} - a_n$  は 7 で割り切れることを示せ。
- (2)  $n$  が 6 の倍数のとき、 $a_n$  は 7 で割り切れないことを示せ。
- (3)  $a_n$  が 7 で割り切れるための  $n$  の条件を求めよ。

$$\begin{aligned} \text{① } a_{n+6} - a_n &= (2^{n+6} + 3^{n+6} + 1) - (2^n + 3^n + 1) \\ &= 63 \cdot 2^n + 728 \cdot 3^n \\ &= 7 \cdot (9 \cdot 2^n + 104 \cdot 3^n) \text{ ㉒} \end{aligned}$$

②  $n=6$  のとき

$$\begin{aligned} a_6 &= 2^6 + 3^6 + 1 \\ &= 64 + 729 + 1 \\ &= 794 = 7 \cdot 113 + 5 \end{aligned}$$

よって 7 で割り切れない。

③ よって  $a_{n+6}$  と  $a_n$  の差を 7 で割った余りは等しいから

$a_1, a_7, \dots, a_{6k-5}$  は 7 で割った余りが 5 であり

$a_2, a_8, \dots, a_{6k-4}$  は

$$\begin{aligned} \text{③ } a_1 &= 5, a_2 = 14, a_3 = 28, a_4 = 48, a_5 = 75, a_6 = 113 + 5 \\ &= 118 \end{aligned}$$

$a_{n+6}$  と  $a_n$  の差を 7 で割った余りは等しいから、

$$a_n = 6k - 4, 6k - 2 \text{ (} k = 1, 2, \dots \text{)} \text{ のとき ㉓}$$

91 [2011 岡山大]

数列  $\{a_n\}$  が次のように帰納的に定められている。

$$a_1 = 0, \quad a_{n+1} = \begin{cases} 2a_n & (n \text{ が奇数のとき}) \\ a_n + 1 & (n \text{ が偶数のとき}) \end{cases} \quad (n = 1, 2, 3, \dots)$$

- (1)  $a_{10}$  を求めよ。
- (2)  $n$  が奇数の場合と偶数の場合それぞれについて、 $a_{n+4}$  を  $a_n$  で表せ。
- (3)  $a_n$  を 3 で割ったときの余りを求めよ。

$$\text{① } a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 2, a_5 = 4, a_6 = 6, a_7 = 7, a_8 = 14, a_9 = 15, a_{10} = 30$$

②  $n$  が奇数のとき

$$a_{n+4} = a_{n+3} + 1 = 2a_{n+2} + 1 = 2(a_{n+1} + 1) + 1 = 2a_{n+1} + 3 = 4a_n + 3$$

$n$  が偶数のとき

$$a_{n+4} = 2a_{n+3} = 2(a_{n+2} + 1) = 2a_{n+2} + 2 = 4a_{n+1} + 6 = 4(a_n + 1) + 6 = 4a_n + 10$$

③ ㉓より

$$n \text{ が奇数のとき } a_{n+4} - a_n = 4a_n + 3 - a_n = 3a_n + 3$$

$$\text{偶数 } a_{n+4} - a_n = 4a_n + 10 - a_n = 3a_n + 10$$

$$a_{n+4} - a_n \text{ が } 3 \text{ の倍数であるから}$$

よって  $a_{n+4}$  と  $a_n$  は 3 の倍数の差になる。

$$a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 2$$

$$a_n \text{ は } 3 \text{ の倍数または } 2 \text{ の倍数}$$

$$n = 4k - 3, 4k - 2 \text{ (} k = 1, 2, \dots \text{)} \text{ のとき } 0$$

$$n = 4k - 1 \text{ " } 1$$

$$n = 4k \text{ " } 2$$

90 [2005 専修大]

一般項が  $a_n = 10^{2n} - 1$  で表される数列について

- (1)  $a_1$  および  $a_5$  を求めよ。
- (2) すべての自然数  $k$  について、 $a_{k+1} - a_k$  が 11 で割り切れることを示せ。
- (3) すべての自然数  $n$  について、 $a_n$  は 11 で割り切れることを数学的帰納法を用いて証明せよ。

92 [2011 首都大学東京]

2つの数列  $\{a_n\}$ ,  $\{b_n\}$  が次の漸化式で与えられているとする。

$$\begin{cases} a_1=4, b_1=3 \\ a_{n+1}=4a_n-3b_n \quad (n=1, 2, 3, \dots) \\ b_{n+1}=3a_n+4b_n \quad (n=1, 2, 3, \dots) \end{cases}$$

- $a_2, a_3, a_4, b_2, b_3, b_4$  を求めよ。
- $a_{n+1}-a_n$  ( $n=1, 2, 3, \dots$ ),  $b_{n+1}-b_n$  ( $n=1, 2, 3, \dots$ ) はともに5の倍数であることを証明せよ。
- $a_n$  ( $n=1, 2, 3, \dots$ ) も  $b_n$  ( $n=1, 2, 3, \dots$ ) も5の倍数ではないことを証明せよ。

$\forall 1) a_2=7, a_3=24, a_4=-14, b_2=11, b_3=-27, b_4=396, \dots$

②)  $a_{n+1}-a_n = 4a_n - 3b_n - a_n = 3a_n - 3b_n = 3(a_n - b_n)$   
 $= 3(4a_{n-1} - 3b_{n-1}) - 3(3a_{n-1} + 4b_{n-1}) - a_n$   
 $= 3(4a_{n-1} - 3b_{n-1}) - 9a_{n-1} - 12b_{n-1} - a_n$   
 $= -5a_{n-1} - 15b_{n-1} - a_n$   
 $= -5(4a_{n-2} - 3b_{n-2}) - 15(3a_{n-2} + 4b_{n-2}) - a_n$   
 $= -52a_{n-2} - 60b_{n-2} - a_n$

③)  $a_n = 1$  とし

$a_n - a_{n-1} = -9 \mid 6 - 4 = -9 \mid 20, b_n - b_{n-1} = -29$  より  $\forall n \geq 2$

$\forall n \geq 2, a_n - a_{n-1} = -9 \mid 6 - 4 = -9 \mid 20, b_n - b_{n-1} = -29$  より  $\forall n \geq 2$

$a_n - a_{n-1} = 5N_1, b_n - b_{n-1} = 5N_2 \quad (N_1, N_2: \text{整数})$

$a_n = 4 + 5k, b_n = 3 + 5l$

$a_{n+1} - a_n = 4a_n - 3b_n - a_n = 3a_n - 3b_n = 3(4a_n - 3b_n)$   
 $= 3(4(4a_{n-1} + 5k) - 3(3a_{n-1} + 5l)) = 3(16a_{n-1} + 20k - 9a_{n-1} - 15l)$   
 $= 3(7a_{n-1} + 20k - 15l) = 3(7a_{n-1} + 5(4k - 3l))$

$b_{n+1} - b_n = 3a_n + 4b_n - b_n = 3a_n + 3b_n = 3(a_n + b_n)$   
 $= 3(4a_{n-1} + 5k + 3a_{n-1} + 5l) = 3(7a_{n-1} + 5(k+l))$   
 $= 3(7a_{n-1} + 5N_3)$

よって  $\forall n \geq 2$

$a_1, b_1$  から  $a_n, b_n$  の性質を帰納法で示す

帰納法の手本

- ③) ②より  $a_{n+1}$  と  $a_n, b_{n+1}$  と  $b_n$  は互いに素な素数で割り切れない

$a_1=4, a_2=7, a_3=-14, a_4=-52, a_5=-9116$

$b_1=3, b_2=11, b_3=-27, b_4=396, b_5=-227$

よって  $n \geq 2$  のとき  $a_n, b_n$  は互いに素な素数で割り切れない

帰納法の手本

94 [岡山大]

数列  $\{a_n\}$  を,  $a_1=1, a_{n+1}=(a_n)^2+1$  ( $n=1, 2, 3, \dots$ ) で定める。

- $a_2, a_3, a_4, a_5, a_6$  を求めよ。
- 自然数  $m$  に対して,  $a_{3m}$  は5で割り切れることを証明せよ。

$\forall 1) a_2=2, a_3=5, a_4=26, a_5=677, a_6=458330$

②)  $a_n = 1$  とし

$a_3=5$  より  $\forall n \geq 1$

$\forall n \geq 1, a_{3n} = 5 \mid a_{3n} = (a_{3n-1})^2 + 1$  と仮定して

$a_{3n} = 5N$  ( $N: \text{整数}$ )

$a_{3n} = 5 \mid a_{3n} = (a_{3n-1})^2 + 1$

$a_{3n} = (a_{3n-1})^2 + 1$

$= (a_{3n-1}^2 + 1)^2 + 1$

$= ((a_{3n-1})^2 + 1)^2 + 1$

$= (a_{3n-1}^2 + 2a_{3n-1} + 2)^2 + 1$

$= (5N^2 + 2)^2 + 1$  ( $N: \text{整数}$ )

$= 5(5N^2 + 4N^2 + 1)$  よって  $\forall n \geq 1$

$a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_{15}, a_{17}, a_{19}, a_{21}, a_{23}, a_{25}, a_{27}, a_{29}, a_{31}, a_{33}, a_{35}, a_{37}, a_{39}, a_{41}, a_{43}, a_{45}, a_{47}, a_{49}, a_{51}, a_{53}, a_{55}, a_{57}, a_{59}, a_{61}, a_{63}, a_{65}, a_{67}, a_{69}, a_{71}, a_{73}, a_{75}, a_{77}, a_{79}, a_{81}, a_{83}, a_{85}, a_{87}, a_{89}, a_{91}, a_{93}, a_{95}, a_{97}, a_{99}, a_{101}, a_{103}, a_{105}, a_{107}, a_{109}, a_{111}, a_{113}, a_{115}, a_{117}, a_{119}, a_{121}, a_{123}, a_{125}, a_{127}, a_{129}, a_{131}, a_{133}, a_{135}, a_{137}, a_{139}, a_{141}, a_{143}, a_{145}, a_{147}, a_{149}, a_{151}, a_{153}, a_{155}, a_{157}, a_{159}, a_{161}, a_{163}, a_{165}, a_{167}, a_{169}, a_{171}, a_{173}, a_{175}, a_{177}, a_{179}, a_{181}, a_{183}, a_{185}, a_{187}, a_{189}, a_{191}, a_{193}, a_{195}, a_{197}, a_{199}, a_{201}, a_{203}, a_{205}, a_{207}, a_{209}, a_{211}, a_{213}, a_{215}, a_{217}, a_{219}, a_{221}, a_{223}, a_{225}, a_{227}, a_{229}, a_{231}, a_{233}, a_{235}, a_{237}, a_{239}, a_{241}, a_{243}, a_{245}, a_{247}, a_{249}, a_{251}, a_{253}, a_{255}, a_{257}, a_{259}, a_{261}, a_{263}, a_{265}, a_{267}, a_{269}, a_{271}, a_{273}, a_{275}, a_{277}, a_{279}, a_{281}, a_{283}, a_{285}, a_{287}, a_{289}, a_{291}, a_{293}, a_{295}, a_{297}, a_{299}, a_{301}, a_{303}, a_{305}, a_{307}, a_{309}, a_{311}, a_{313}, a_{315}, a_{317}, a_{319}, a_{321}, a_{323}, a_{325}, a_{327}, a_{329}, a_{331}, a_{333}, a_{335}, a_{337}, a_{339}, a_{341}, a_{343}, a_{345}, a_{347}, a_{349}, a_{351}, a_{353}, a_{355}, a_{357}, a_{359}, a_{361}, a_{363}, a_{365}, a_{367}, a_{369}, a_{371}, a_{373}, a_{375}, a_{377}, a_{379}, a_{381}, a_{383}, a_{385}, a_{387}, a_{389}, a_{391}, a_{393}, a_{395}, a_{397}, a_{399}, a_{401}, a_{403}, a_{405}, a_{407}, a_{409}, a_{411}, a_{413}, a_{415}, a_{417}, a_{419}, a_{421}, a_{423}, a_{425}, a_{427}, a_{429}, a_{431}, a_{433}, a_{435}, a_{437}, a_{439}, a_{441}, a_{443}, a_{445}, a_{447}, a_{449}, a_{451}, a_{453}, a_{455}, a_{457}, a_{459}, a_{461}, a_{463}, a_{465}, a_{467}, a_{469}, a_{471}, a_{473}, a_{475}, a_{477}, a_{479}, a_{481}, a_{483}, a_{485}, a_{487}, a_{489}, a_{491}, a_{493}, a_{495}, a_{497}, a_{499}, a_{501}, a_{503}, a_{505}, a_{507}, a_{509}, a_{511}, a_{513}, a_{515}, a_{517}, a_{519}, a_{521}, a_{523}, a_{525}, a_{527}, a_{529}, a_{531}, a_{533}, a_{535}, a_{537}, a_{539}, a_{541}, a_{543}, a_{545}, a_{547}, a_{549}, a_{551}, a_{553}, a_{555}, a_{557}, a_{559}, a_{561}, a_{563}, a_{565}, a_{567}, a_{569}, a_{571}, a_{573}, a_{575}, a_{577}, a_{579}, a_{581}, a_{583}, a_{585}, a_{587}, a_{589}, a_{591}, a_{593}, a_{595}, a_{597}, a_{599}, a_{601}, a_{603}, a_{605}, a_{607}, a_{609}, a_{611}, a_{613}, a_{615}, a_{617}, a_{619}, a_{621}, a_{623}, a_{625}, a_{627}, a_{629}, a_{631}, a_{633}, a_{635}, a_{637}, a_{639}, a_{641}, a_{643}, a_{645}, a_{647}, a_{649}, a_{651}, a_{653}, a_{655}, a_{657}, a_{659}, a_{661}, a_{663}, a_{665}, a_{667}, a_{669}, a_{671}, a_{673}, a_{675}, a_{677}, a_{679}, a_{681}, a_{683}, a_{685}, a_{687}, a_{689}, a_{691}, a_{693}, a_{695}, a_{697}, a_{699}, a_{701}, a_{703}, a_{705}, a_{707}, a_{709}, a_{711}, a_{713}, a_{715}, a_{717}, a_{719}, a_{721}, a_{723}, a_{725}, a_{727}, a_{729}, a_{731}, a_{733}, a_{735}, a_{737}, a_{739}, a_{741}, a_{743}, a_{745}, a_{747}, a_{749}, a_{751}, a_{753}, a_{755}, a_{757}, a_{759}, a_{761}, a_{763}, a_{765}, a_{767}, a_{769}, a_{771}, a_{773}, a_{775}, a_{777}, a_{779}, a_{781}, a_{783}, a_{785}, a_{787}, a_{789}, a_{791}, a_{793}, a_{795}, a_{797}, a_{799}, a_{801}, a_{803}, a_{805}, a_{807}, a_{809}, a_{811}, a_{813}, a_{815}, a_{817}, a_{819}, a_{821}, a_{823}, a_{825}, a_{827}, a_{829}, a_{831}, a_{833}, a_{835}, a_{837}, a_{839}, a_{841}, a_{843}, a_{845}, a_{847}, a_{849}, a_{851}, a_{853}, a_{855}, a_{857}, a_{859}, a_{861}, a_{863}, a_{865}, a_{867}, a_{869}, a_{871}, a_{873}, a_{875}, a_{877}, a_{879}, a_{881}, a_{883}, a_{885}, a_{887}, a_{889}, a_{891}, a_{893}, a_{895}, a_{897}, a_{899}, a_{901}, a_{903}, a_{905}, a_{907}, a_{909}, a_{911}, a_{913}, a_{915}, a_{917}, a_{919}, a_{921}, a_{923}, a_{925}, a_{927}, a_{929}, a_{931}, a_{933}, a_{935}, a_{937}, a_{939}, a_{941}, a_{943}, a_{945}, a_{947}, a_{949}, a_{951}, a_{953}, a_{955}, a_{957}, a_{959}, a_{961}, a_{963}, a_{965}, a_{967}, a_{969}, a_{971}, a_{973}, a_{975}, a_{977}, a_{979}, a_{981}, a_{983}, a_{985}, a_{987}, a_{989}, a_{991}, a_{993}, a_{995}, a_{997}, a_{999}, a_{1001}, a_{1003}, a_{1005}, a_{1007}, a_{1009}, a_{1011}, a_{1013}, a_{1015}, a_{1017}, a_{1019}, a_{1021}, a_{1023}, a_{1025}, a_{1027}, a_{1029}, a_{1031}, a_{1033}, a_{1035}, a_{1037}, a_{1039}, a_{1041}, a_{1043}, a_{1045}, a_{1047}, a_{1049}, a_{1051}, a_{1053}, a_{1055}, a_{1057}, a_{1059}, a_{1061}, a_{1063}, a_{1065}, a_{1067}, a_{1069}, a_{1071}, a_{1073}, a_{1075}, a_{1077}, a_{1079}, a_{1081}, a_{1083}, a_{1085}, a_{1087}, a_{1089}, a_{1091}, a_{1093}, a_{1095}, a_{1097}, a_{1099}, a_{1101}, a_{1103}, a_{1105}, a_{1107}, a_{1109}, a_{1111}, a_{1113}, a_{1115}, a_{1117}, a_{1119}, a_{1121}, a_{1123}, a_{1125}, a_{1127}, a_{1129}, a_{1131}, a_{1133}, a_{1135}, a_{1137}, a_{1139}, a_{1141}, a_{1143}, a_{1145}, a_{1147}, a_{1149}, a_{1151}, a_{1153}, a_{1155}, a_{1157}, a_{1159}, a_{1161}, a_{1163}, a_{1165}, a_{1167}, a_{1169}, a_{1171}, a_{1173}, a_{1175}, a_{1177}, a_{1179}, a_{1181}, a_{1183}, a_{1185}, a_{1187}, a_{1189}, a_{1191}, a_{1193}, a_{1195}, a_{1197}, a_{1199}, a_{1201}, a_{1203}, a_{1205}, a_{1207}, a_{1209}, a_{1211}, a_{1213}, a_{1215}, a_{1217}, a_{1219}, a_{1221}, a_{1223}, a_{1225}, a_{1227}, a_{1229}, a_{1231}, a_{1233}, a_{1235}, a_{1237}, a_{1239}, a_{1241}, a_{1243}, a_{1245}, a_{1247}, a_{1249}, a_{1251}, a_{1253}, a_{1255}, a_{1257}, a_{1259}, a_{1261}, a_{1263}, a_{1265}, a_{1267}, a_{1269}, a_{1271}, a_{1273}, a_{1275}, a_{1277}, a_{1279}, a_{1281}, a_{1283}, a_{1285}, a_{1287}, a_{1289}, a_{1291}, a_{1293}, a_{1295}, a_{1297}, a_{1299}, a_{1301}, a_{1303}, a_{1305}, a_{1307}, a_{1309}, a_{1311}, a_{1313}, a_{1315}, a_{1317}, a_{1319}, a_{1321}, a_{1323}, a_{1325}, a_{1327}, a_{1329}, a_{1331}, a_{1333}, a_{1335}, a_{1337}, a_{1339}, a_{1341}, a_{1343}, a_{1345}, a_{1347}, a_{1349}, a_{1351}, a_{1353}, a_{1355}, a_{1357}, a_{1359}, a_{1361}, a_{1363}, a_{1365}, a_{1367}, a_{1369}, a_{1371}, a_{1373}, a_{1375}, a_{1377}, a_{1379}, a_{1381}, a_{1383}, a_{1385}, a_{1387}, a_{1389}, a_{1391}, a_{1393}, a_{1395}, a_{1397}, a_{1399}, a_{1401}, a_{1403}, a_{1405}, a_{1407}, a_{1409}, a_{1411}, a_{1413}, a_{1415}, a_{1417}, a_{1419}, a_{1421}, a_{1423}, a_{1425}, a_{1427}, a_{1429}, a_{1431}, a_{1433}, a_{1435}, a_{1437}, a_{1439}, a_{1441}, a_{1443}, a_{1445}, a_{1447}, a_{1449}, a_{1451}, a_{1453}, a_{1455}, a_{1457}, a_{1459}, a_{1461}, a_{1463}, a_{1465}, a_{1467}, a_{1469}, a_{1471}, a_{1473}, a_{1475}, a_{1477}, a_{1479}, a_{1481}, a_{1483}, a_{1485}, a_{1487}, a_{1489}, a_{1491}, a_{1493}, a_{1495}, a_{1497}, a_{1499}, a_{1501}, a_{1503}, a_{1505}, a_{1507}, a_{1509}, a_{1511}, a_{1513}, a_{1515}, a_{1517}, a_{1519}, a_{1521}, a_{1523}, a_{1525}, a_{1527}, a_{1529}, a_{1531}, a_{1533}, a_{1535}, a_{1537}, a_{1539}, a_{1541}, a_{1543}, a_{1545}, a_{1547}, a_{1549}, a_{1551}, a_{1553}, a_{1555}, a_{1557}, a_{1559}, a_{1561}, a_{1563}, a_{1565}, a_{1567}, a_{1569}, a_{1571}, a_{1573}, a_{1575}, a_{1577}, a_{1579}, a_{1581}, a_{1583}, a_{1585}, a_{1587}, a_{1589}, a_{1591}, a_{1593}, a_{1595}, a_{1597}, a_{1599}, a_{1601}, a_{1603}, a_{1605}, a_{1607}, a_{1609}, a_{1611}, a_{1613}, a_{1615}, a_{1617}, a_{1619}, a_{1621}, a_{1623}, a_{1625}, a_{1627}, a_{1629}, a_{1631}, a_{1633}, a_{1635}, a_{1637}, a_{1639}, a_{1641}, a_{1643}, a_{1645}, a_{1647}, a_{1649}, a_{1651}, a_{1653}, a_{1655}, a_{1657}, a_{1659}, a_{1661}, a_{1663}, a_{1665}, a_{1667}, a_{1669}, a_{1671}, a_{1673}, a_{1675}, a_{1677}, a_{1679}, a_{1681}, a_{1683}, a_{1685}, a_{1687}, a_{1689}, a_{1691}, a_{1693}, a_{1695}, a_{1697}, a_{1699}, a_{1701}, a_{1703}, a_{1705}, a_{1707}, a_{1709}, a_{1711}, a_{1713}, a_{1715}, a_{1717}, a_{1719}, a_{1721}, a_{1723}, a_{1725}, a_{1727}, a_{1729}, a_{1731}, a_{1733}, a_{1735}, a_{1737}, a_{1739}, a_{1741}, a_{1743}, a_{1745}, a_{1747}, a_{1749}, a_{1751}, a_{1753}, a_{1755}, a_{1757}, a_{1759}, a_{1761}, a_{1763}, a_{1765}, a_{1767}, a_{1769}, a_{1771}, a_{1773}, a_{1775}, a_{1777}, a_{1779}, a_{1781}, a_{1783}, a_{1785}, a_{1787}, a_{1789}, a_{1791}, a_{1793}, a_{1795}, a_{1797}, a_{1799}, a_{1801}, a_{1803}, a_{1805}, a_{1807}, a_{1809}, a_{1811}, a_{1813}, a_{1815}, a_{1817}, a_{1819}, a_{1821}, a_{1823}, a_{1825}, a_{1827}, a_{1829}, a_{1831}, a_{1833}, a_{1835}, a_{1837}, a_{1839}, a_{1841}, a_{1843}, a_{1845}, a_{1847}, a_{1849}, a_{1851}, a_{1853}, a_{1855}, a_{1857}, a_{1859}, a_{1861}, a_{1863}, a_{1865}, a_{1867}, a_{1869}, a_{1871}, a_{1873}, a_{1875}, a_{1877}, a_{1879}, a_{1881}, a_{1883}, a_{1885}, a_{1887}, a_{1889}, a_{1891}, a_{1893}, a_{1895}, a_{1897}, a_{1899}, a_{1901}, a_{1903}, a_{1905}, a_{1907}, a_{1909}, a_{1911}, a_{1913}, a_{1915}, a_{1917}, a_{1919}, a_{1921}, a_{1923}, a_{1925}, a_{1927}, a_{1929}, a_{1931}, a_{1933}, a_{1935}, a_{1937}, a_{1939}, a_{1941}, a_{1943}, a_{1945}, a_{1947}, a_{1949}, a_{1951}, a_{1953}, a_{1955}, a_{1957}, a_{1959}, a_{1961}, a_{1963}, a_{1965}, a_{1967}, a_{1969}, a_{1971}, a_{1973}, a_{1975}, a_{1977}, a_{1979}, a_{1981}, a_{1983}, a_{1985}, a_{1987}, a_{1989}, a_{1991}, a_{1993}, a_{1995}, a_{1997}, a_{1999}, a_{2001}, a_{2003}, a_{2005}, a_{2007}, a_{2009}, a_{2011}, a_{2013}, a_{2015}, a_{2017}, a_{2019}, a_{2021}, a_{2023}, a_{2025}, a_{2027}, a_{2029}, a_{2031}, a_{2033}, a_{2035}, a_{2037}, a_{2039}, a_{2041}, a_{2043}, a_{2045}, a_{2047}, a_{2049}, a_{2051}, a_{2053}, a_{2055}, a_{2057}, a_{2059}, a_{2061}, a_{2063}, a_{2065}, a_{2067}, a_{2069}, a_{2071}, a_{2073}, a_{2075}, a_{2077}, a_{2079}, a_{2081}, a_{2083}, a_{2085}, a_{2087}, a_{2089}, a_{2091}, a_{2093}, a_{2095}, a_{2097}, a_{2099}, a_{2101}, a_{2103}, a_{2105}, a_{2107}, a_{2109}, a_{2111}, a_{2113}, a_{2115}, a_{2117}, a_{2119}, a_{2121}, a_{2123}, a_{2125}, a_{2127}, a_{2129}, a_{2131}, a_{2133}, a_{2135}, a_{2137}, a_{2139}, a_{2141}, a_{2143}, a_{2145}, a_{2147}, a_{2149}, a_{2151}, a_{2153}, a_{2155}, a_{2157}, a_{2159}, a_{2161}, a_{2163}, a_{2165}, a_{2167}, a_{2169}, a_{2171}, a_{2173}, a_{2175}, a_{2177}, a_{2179}, a_{2181}, a_{2183}, a_{2185}, a_{2187}, a_{2189}, a_{2191}, a_{2193}, a_{2195}, a_{2197}, a_{2199}, a_{2201}, a_{2203}, a_{2205}, a_{2207}, a_{2209}, a_{2211}, a_{2213}, a_{2215}, a_{2217}, a_{2219}, a_{2221}, a_{2223}, a_{2225}, a_{2227}, a_{2229}, a_{2231}, a_{2233}, a_{2235}, a_{2237}, a_{2239}, a_{2241}, a_{2243}, a_{2245}, a_{2247}, a_{2249}, a_{2251}, a_{2253}, a_{2255}, a_{2257}, a_{2259}, a_{2261}, a_{2263}, a_{2265}, a_{2267}, a_{2269}, a_{2271}, a_{2273}, a_{2275}, a_{2277}, a_{2279}, a_{2281}, a_{2283}, a_{2285}, a_{2287}, a_{2289}, a_{2291}, a_{2293}, a_{2295}, a_{2297}, a_{2299}, a_{2301}, a_{2303}, a_{2305}, a_{2307}, a_{2309}, a_{2311}, a_{2313}, a_{2315}, a_{2317}, a_{2319}, a_{2321}, a_{2323}, a_{2325}, a_{2327}, a_{2329}, a_{2331}, a_{2333}, a_{2335}, a_{2337}, a_{2339}, a_{2341}, a_{2343}, a_{2345}, a_{2347}, a_{2349}, a_{2351}, a_{2353}, a_{2355}, a_{2357}, a_{2359}, a_{2361}, a_{2363}, a_{2365}, a_{2367}, a_{2369}, a_{2371}, a_{2373}, a_{2375}, a_{2377}, a_{2379}, a_{2381}, a_{2383}, a_{2385}, a_{2387}, a_{2389}, a_{2391}, a_{2393}, a_{2395}, a_{2397}, a_{2399}, a_{2401}, a_{2403}, a_{2405}, a_{2407}, a_{2409}, a_{2411}, a_{2413}, a_{2415}, a_{2417}, a_{2419}, a_{2421}, a_{2423}, a_{2425}, a_{2427}, a_{2429}, a_{2431}, a_{2433}, a_{2435}, a_{2437}, a_{2439}, a_{2441}, a_{2443}, a_{2445}, a_{2447}, a_{2449}, a_{2451}, a_{2453}, a_{2455}, a_{2457}, a_{2459}, a_{2461}, a_{2463}, a_{2465}, a_{2467}, a_{2469}, a_{2471}, a_{2473}, a_{2475}, a_{2477}, a_{2479}, a_{2481}, a_{2483}, a_{2485}, a_{2487}, a_{2489}, a_{2491}, a_{2493}, a_{2495}, a_{2497}, a_{2499}, a_{2501}, a_{2503}, a_{2505}, a_{2507}, a_{2509}, a_{2511}, a_{2513}, a_{2515}, a_{2517}, a_{2519}, a_{2521}, a_{2523}, a_{2525}, a_{2527}, a_{2529}, a_{2531}, a_{2533}, a_{2535}, a_{2537}, a_{2539}, a_{2541}, a_{2543}, a_{2545}, a_{2547}, a_{2549}, a_{2551}, a_{2553}, a_{2555}, a_{2557}, a_{2559}, a_{2561}, a_{2563}, a_{2565}, a_{2567}, a_{2569}, a_{2571}, a_{2573}, a_{2575}, a_{2577}, a_{2579}, a_{2581}, a_{2583}, a_{2585}, a_{2587}, a_{2589}, a_{2591}, a_{2593}, a_{2595}, a_{2597}, a_{2599}, a_{2601}, a_{2603}, a_{2605}, a_{2607}, a_{2609}, a_{2611}, a_{2613}, a_{2615}, a_{2617}, a_{2619}, a_{2621}, a_{2623}, a_{2625}, a_{2627}, a_{2629}, a_{2631}, a_{2633}, a_{2635}, a_{2637}, a_{2639}, a_{2641}, a_{2643}, a_{2645}, a_{2647}, a_{2649}, a_{2651}, a_{2653}, a_{2655}, a_{2657}, a_{2659}, a_{2661}, a_{2663}, a_{2665}, a_{2667}, a_{2669}, a_{2671}, a_{2673}, a_{2675}, a_{2677}, a_{2679}, a_{2681}, a_{2683}, a_{2685}, a_{2687}, a_{2689}, a_{2691}, a_{2693}, a_{2695}, a_{2697}, a_{2699}, a_{2701}, a_{2703}, a_{2705}, a_{2707}, a_{2709}, a_{2711}, a_{2713}, a_{2715}, a_{2717}, a_{2719}, a_{2721}, a_{2723}, a_{2725}, a_{2727}, a_{2729}, a_{2731}, a_{2733}, a_{2735}, a_{2737}, a_{2739}, a_{2741}, a_{2743}, a_{2745}, a_{2747}, a_{2749}, a_{275$

97 [2010 県立広島大]

数列  $\{a_n\}$  を

$$a_1=1, a_2=1, a_{n+2}=7a_{n+1}+a_n \quad (n=1, 2, 3, \dots)$$

によって定める。

- (1)  $a_{n+3}$  を  $a_n, a_{n+1}$  で表せ。
- (2)  $a_{3n}$  ( $n=1, 2, 3, \dots$ ) が偶数であることを数学的帰納法で証明せよ。
- (3)  $a_{3n}$  ( $n=1, 2, 3, \dots$ ) が3の倍数となることを示せ。

①  $a_{n+3} = 7a_{n+2} + a_{n+1}$   
 $= 7(7a_{n+1} + a_n) + a_{n+1}$   
 $= 50a_{n+1} + 7a_n$

②  $a_1=1, a_2=1, a_3=8, a_4=57$   
 $a_5=392$  より成り立つ

③  $a_n = f_k$  ( $k=1, 2, \dots$ )  $a$  と  $b$  を用いて表す

$$a_{2k} = 2N_k \quad (N_k: \text{整数})$$

$$a_n = f_k \quad (a \text{ と } b \text{ を用いて表す})$$

$$a_{3k+3} = 50a_{3k+1} + 7a_{3k}$$

$$= 2(25a_{3k+1} + 7N_k) \quad (a_{3k+1} \text{ は } 2 \text{ の倍数, } N_k \text{ は整数})$$

よって成り立つ

④  $a_{3n}$  が偶数であることを帰納法より

証明する。小問 ②

⑤  $a_1=1, a_2=1$

$$a_3=8, a_4=57 \text{ より成り立つ}$$

⑥  $a_n = f_k$  ( $k=1, 2, \dots$ )  $a$  と  $b$  を用いて表す

$$a_{2k} = 2N_k \quad (N_k: \text{整数})$$

$$a_n = f_k \quad (a \text{ と } b \text{ を用いて表す})$$

$$a_{4k+4} = 50a_{4k+2} + 7a_{4k+1}$$

$$= 2(25a_{4k+2} + 7a_{4k+1})$$

$$= 2(19a_{4k+1} + 50N_k)$$

よって成り立つ

⑦  $a_{3n}$  が偶数であることを帰納法より

証明する。小問 ②

99 [1993 東京大]

整数からなる数列  $\{a_n\}$  を漸化式  $a_1=1, a_2=3, a_{n+2}=3a_{n+1}-7a_n$  で定める。  $a_n$  が偶数となる  $n$  を決定せよ。

$$a_3=2, a_4=-16, a_5=-59, a_6=-72, \dots \quad \text{奇数項は } 1: a_{2k+1} \text{ と } 2: a_{2k}$$

$a_{2k+1} - a_{2k}$  の偶数であることを示す

$$a_{2k+1} = |a| \text{ と } 3$$

$$a_k - a_{k-1} = -15 - 1 = -16 \text{ より成り立つ}$$

②  $a_n = f_k$   $a$  と  $b$  を用いて表す

$$a_{2k+2} - a_{2k} = 2N \quad (N: \text{整数})$$

$$a_n = f_k \quad (a \text{ と } b \text{ を用いて表す})$$

$$a_{2k+2} - a_{2k+1} = 3a_{2k+1} - 7a_{2k} - a_{2k+1}$$

$$= 2a_{2k+1} - 7a_{2k} - a_{2k+1}$$

$$= 3a_{2k+1} - 2a_{2k+1} + 4a_{2k}$$

$$= 3(a_{2k+1} - a_{2k}) - 2(11a_{2k+1} - 26a_{2k})$$

$$= 2(311 - 11a_{2k+1} + 26a_{2k})$$

よって成り立つ

③  $a_{2k}$  が偶数であることを帰納法より

証明する。小問 ②

$a_{2k+1} - a_{2k}$  は偶数であることを示す

$a_{2k}$  と  $a_{2k+1}$  の偶数であることを示す

$$a_1=1, a_2=3, a_3=2 \text{ より}$$

$a_{2k}$  が偶数であることを帰納法より

証明する。小問 ②

以下、2を3で割ることを示す

$$a_{2k+2} = 3a_{2k+1} - 7a_{2k}$$

$$\therefore a_{2k+2} \equiv 3a_{2k+1} - 7a_{2k} \pmod{2}$$

$$\therefore a_{2k+2} \equiv a_{2k+1} - a_{2k} \pmod{2}$$

$$a_1 \equiv 1, a_2 \equiv 1$$

$$a_3 \equiv a_2 - a_1 \equiv 1 - 1 \equiv 0$$

$$a_4 \equiv a_3 - a_2 \equiv 1 - 1 \equiv 0$$

$$a_5 \equiv a_4 - a_3 \equiv 1 - 1 \equiv 0$$

$$a_6 \equiv a_5 - a_4 \equiv 1 - 1 \equiv 0$$

$$\vdots$$

よって  $2 \mid a_n$  となる  $n$  は  $3$  の倍数であることを示す。

98 [大阪府立大]

数列  $\{a_n\}$  を  $a_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right)$  で定める。

- (1)  $a_2, a_3, a_4$  を求めよ。
- (2)  $a_{n+2} = a_{n+1} + a_n$  を示せ。
- (3) 任意の自然数  $n$  に対して、 $a_n$  は整数であり、 $a_{3n}$  は3の倍数であることを示せ。

100 [熊本大]

- $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$  ( $n \geq 3$ ) により定まる数列  $\{a_n\}$  について、
- (1)  $n=3, 4, \dots, 9$  に対して  $a_n$  の値を求めよ。
  - (2)  $n$  が3の倍数ならば  $a_n$  は偶数であり、 $n$  が3の倍数でなければ  $a_n$  は奇数であることを示せ。